

BRNO UNIVERSITY OF TECHNOLOGY

Faculty of Mechanical Engineering

Institute of mechanics of solids, mechatronics and bimechanics

Lecture notes

COMPUTATIONAL MECHANICS I

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1. INTRODUCTION

These course have been written for the students in the 4th year (1st year of 2nd stage) on the faculty of mechanical engineering, specialization *Computer Mechanics I*. In general, the text may be used for all students studying the theory of vibrations.

Numerical analysis has been studied since before the time of Newton. These studies were concerned with numerical procedures for approximating solutions to problems that could not be conveniently solved by theoretical methods leading to analytical solutions expressed by formulas. Until the arrival of high-speed computing machines, such methods were difficult to use, and their full potential could not be realized.

With the arrival of modern computing machines the whole character of numerical analysis has changed. Iterative methods can now be used with much greater ease and effectiveness. Very large linear systems can be studied numerically. Numerical solutions of differential equations can be obtained using very small step size, or even variable step size, with thousands of steps. Today, the elaboration of efficient computational models for the analysis of the dynamic behaviour of machines and structures has become a routine task. Computer can be used to automate many engineering applications. When they are used effectively they produce results that demonstrate an increase in productivity and a reduction in numerical errors. Many professional software are used, like ANSYS, NASTRAN, SYSNOIS, RAYNOLDS, MAPLE, MATLAB, etc., to solve the analysis of stresses and dynamic loading in frames and machines parts.

In every technical solution we have to observe following elements of solution:

Statement of problem

Determine the effective theoretical approach

Mathematical description of the model

Algorithm development

Input/output design

Choose numerical methods

Computer implementation

Program development

Program testing

We see that the computer implementation is one of all necessary steps only. Therefore in each type of problem we will review the theoretical ideas connected with such problem.

2. ANALYTICAL DYNAMICS OF DISCRETE SYSTEMS

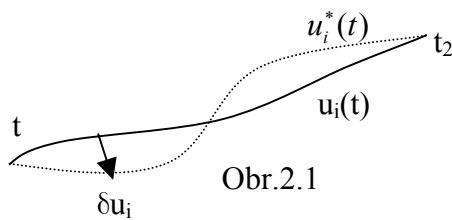
Analytical dynamics is based on principle of virtual work, which is transferred to the concepts to the energy and work. Because these quantities are scalars is such approach to calculus called *Scalar Dynamics* in opposite to the *vector dynamics*. This method provides a very powerful tool for two main reasons:

- It considerably simplifies the analytical formulation of the motion equations for a complex mechanical system
- It gives rise to approximate numerical methods for the solution for both discrete and continuous systems in the most natural manner

2.1 Principle of virtual work for a particle

Let us consider a particle of mass m_i , submitted to a force field \mathbf{X} of components X_i . The dynamic equilibrium of the particle can be expressed in d'Alembert's form

$$m\ddot{u}_i - X_i = 0 \quad i = 1, 2, 3 \quad (2.1.1)$$



where u_i represents the displacement of the particle.

Let us consider that the particle follows during the time interval $[t_1, t_2]$ a motion trajectory u_i^* distinct from the real one u_i (fig. 2.1). This allows us to define the virtual displacement of the particle the

relationship

$$\delta u_i = u_i^* - u_i; \quad \delta u_i(t_1) = \delta u_i(t_2) = 0 \quad (2.1.2)$$

The virtual displacement may be arbitrary in the time interval (t_1, t_2) . We suppose only, that on the beginning and end of the interval is the displacement equal for both paths. Therefore $\delta u_i(t_1) = \delta u_i(t_2) = 0$. From (2.1.2) it is seen, that the change operator δ is connected with the time derivative operator:

$$\frac{d}{dt}(\delta u_i) = \frac{d}{dt}(u_i^* - u_i) = \dot{u}_i^* - \dot{u}_i = \delta \dot{u}_i \quad (2.1.3)$$

If we multiply (2.1.1) by the associated virtual displacement and sum over the components, we get

$$\sum_{i=1}^3 (m\ddot{u}_i - X_i)\delta u_i = 0 \quad (2.1.4)$$

which shows that

The virtual work produced by the effective forces acting on the particle during a virtual displacement δu_i is equal to zero.

If we consider N particles in a system the eq.(2.1.1) will be changed to

$$m_k \ddot{u}_{ik} - X_{ik} = 0 \quad \text{for } i=1,2,3; \quad k=1,\dots,N \quad (2.1.5)$$

the virtual work principle for the system of particles takes the form

$$\sum_{k=1}^N \sum_{i=1}^3 (m_k \ddot{u}_{ik} - X_{ik})\delta u_{ik} = 0, \quad (2.1.6)$$

and it can be stated that

If the virtual work equation is satisfied for any virtual displacement compatible with the kinematical constraints, the system is in dynamic equilibrium.

2.2 The kinematical constraints

Without kinematical constraints, the state of the system would be completely defined by the $3N$ displacements components u_{ik} . They represent the instantaneous configuration. Starting from the reference configuration x_{ik} . It is possible to determine the instantaneous configuration from the equation

$$\xi_{ik}(t) = x_{ik} + u_{ik}(x, t) \quad (2.2.1)$$

The system is said to possess $3N$ degrees of freedom.

In however, the particles are submitted to kinematic constraints which restrain their motion and define dependency relationships between particles. The constraints are divided on:

- a) *Holonomic constraints*, which are defined by relationships of type

$$f(\xi_{ik}, t) = 0 \quad (2.2.2)$$

Every holonomic constraint reduces by one the number of degrees of freedom of the system

If there is not explicit dependence with respect to time, the constraints are said to be *scleronomic*. Otherwise they are called *rheonomic*.

b) *Non-holonomic constraints* are such, if they are not put in the form (2.2.2). In particular, non-holonomic constraints often take the form of differential relationship

$$f(\dot{\xi}_{ik}, \xi_{ik}, t) = 0 \quad (2.2.3)$$

These equations are generally not integrable .

2.3. Generalized coordinates and displacements

If s holonomic constraints exist between the $3N$ displacements of the system, the number of degrees of freedom is then reduced to $3N - s$. It is then necessary to define $n = 3N - s$ configuration parameters, or *generalized coordinates*, noted (q_1, \dots, q_n) in terms of which the displacements of the system of particles are expressed in the form

$$u_{ik}(x, t) = U_{ik}(q_1, q_2, \dots, q_n, t) \quad (2.3.1)$$

When only holonomic constraints are applied to the system, the generalized coordinates are independent and may be varied in any arbitrary manner without violating the kinematic constraints. The virtual displacement compatible with the holonomic constraints may be expressed in the form

$$\delta u_{ik} = \sum_{s=1}^n \frac{\partial U_{ik}}{\partial q_s} \delta q_s \quad (2.3.2)$$

The virtual work equation becomes

$$\sum_{s=1}^n \left[\sum_{k=1}^N \sum_{i=1}^3 (m_k \ddot{u}_{ik} - X_{ik}) \frac{\partial U_{ik}}{\partial q_s} \right] \delta q_s \quad (2.3.3)$$

and putting the second term in the form

$$\sum_{s=1}^n Q_s \delta q_s$$

Q_s is called the *generalized force*

The generalized force conjugated to the degree of freedom appears as

$$Q_s = \sum_{k=1}^N \sum_{i=1}^3 X_{ik} \frac{\partial U_{ik}}{\partial q_s} \quad (2.3.4)$$

The first term in (2.3.3) has the meaning of *generalized inertia force*

2.4. Hamilton's principle for conservative systems

Hamilton's principle is a time integrated form of the virtual work principle obtained by transforming the expression

$$\int_{t_1}^{t_2} \left[\sum_{k=1}^N \sum_{i=1}^3 (-m_k \ddot{u}_{ik} + X_{ik}) \delta u_{ik} \right] dt = 0 \quad (2.4.1)$$

where δu_{ik} are arbitrary but compatible virtual displacements which verify the end conditions (2.1.2).

First, let us assume that the applied forces X_{ik} can be derived from the potential energy, so that virtual work can be expressed in the form

$$\sum_{k=1}^N \sum_{i=1}^3 X_{ik} \delta u_{ik} = \sum_{s=1}^n Q_s \delta q_s = -\delta E_p \quad (2.4.2)$$

The generalized forces are derived from the potential energy by the relationship

$$Q_s = -\frac{\partial E_p}{\partial q_s} \quad (2.4.3)$$

The term associated with inertia forces is transformed by noting that

$$\frac{d}{dt} (m_k \dot{u}_{ik} \delta u_{ik}) = m_k \ddot{u}_{ik} \delta u_{ik} + m_k \dot{u}_{ik} \delta \dot{u}_{ik} = m_k \ddot{u}_{ik} \delta u_{ik} + \frac{1}{2} (m_k \dot{u}_{ik} \dot{u}_{ik})$$

Owing to the definition of the kinetic energy of the system

$$E_k = \frac{1}{2} \sum_{k=1}^N \sum_{i=1}^3 m_k \dot{u}_{ik} \dot{u}_{ik} \quad (2.4.4)$$

the equation (2.4.1) may be written in the form

$$\left[-\sum_{k=1}^N \sum_{i=1}^3 m_k \dot{u}_{ik} \delta u_{ik} \right]_{t_1}^{t_2} + \delta \int_{t_1}^{t_2} (E_k - E_p) dt = 0 \quad (2.4.5)$$

in which the time boundary condition can be eliminated by taking account of the end conditions (2.1.2).

The functional in (2.4.5) can be expressed in terms of generalized coordinates q_s noticing that

$$\dot{u}_{ik} = \frac{\partial U_{ik}}{\partial t} + \sum_{s=1}^n \frac{\partial U_{ik}}{\partial q_s} \dot{q}_s \quad (2.4.6)$$

and therefore

$$E_k = E_k(q, \dot{q}, t) \quad E_p = E_p(q, t) \quad (2.4.7)$$

Using the equations (2.1.2) and (2.3.2) the boundary condition may be also be written

$$\delta q_s(t_1) = \delta q_s(t_2) = 0 \quad (2.4.8)$$

Hamilton's principle for conservative system may thus be stated in the following form

The real trajectory of the system is such as the integral

$$\int_{t_1}^{t_2} (E_k - E_p) dt$$

remains stationary with respect to any compatible virtual displacement arbitrary between both instants t_1 and t_2 but vanishing at the ends of the interval.

$$\boxed{\delta \int_{t_1}^{t_2} (E_k - E_p) dt = 0; \quad \delta q_s(t_1) = \delta q_s(t_2) = 0} \quad (2.4.9)$$

2.5 Lagrange's equations of 2nd order

Starting from expression (2.4.9) the system of equations of motion are easily obtained in terms of generalized coordinates. We can write

$$\delta E_k = \sum_{s=1}^n \left(\frac{\partial E_k}{\partial q_s} \delta q_s + \frac{\partial E_k}{\partial \dot{q}_s} \delta \dot{q}_s \right)$$

Using more explicit form of (2.4.9)

$$\int_{t_1}^{t_2} \sum_{s=1}^n \left[\left(\frac{\partial E_k}{\partial q_s} + Q_s \right) \delta q_s + \frac{\partial E_k}{\partial \dot{q}_s} \delta \dot{q}_s \right] dt = 0$$

The second term can be integrate by parts

$$\int_{t_1}^{t_2} \frac{\partial E_k}{\partial \dot{q}_s} \delta \dot{q}_s dt = \left[\frac{\partial E_k}{\partial \dot{q}_s} \delta q_s \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_s} \right) \delta q_s dt$$

Taking into account the boundary conditions the following is equivalent to Hamilton's principle

$$\int_{t_1}^{t_2} \sum_{s=1}^n \left[-\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_s} \right) + \frac{\partial E_k}{\partial q_s} + Q_s \right] \delta q_s dt \quad (2.5.1)$$

The variation δq_s is arbitrary on the whole interval and the equations of motion result in the form obtained by Lagrange

$$\boxed{\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_s} \right) - \frac{\partial E_k}{\partial q_s} = Q_s \quad s = 1, 2, \dots, n} \quad (2.5.2)$$

2.5.1 Classification of generalized forces

A distinction can be made between *internal* and *external* forces. In both cases they are said to be *conservative* if the associated virtual work is recoverable.

Internal forces

Among the internal forces, the distinction can be made between the linking forces, those associated with elastic deformations and those resulting from a dissipation mechanism.

a. Linking forces

Linking forces appear in a rigid connection between two particles. They are such as the system of forces is in equilibrium

$$X_{i1} + X_{i2} = 0$$

The virtual work associated with the virtual displacement is

$$\delta A = \sum_{i=1}^3 (X_{i1} \delta u_{i1} + X_{i2} \delta u_{i2}) = \sum_{i=1}^3 X_{i1} (\delta u_{i1} - \delta u_{i2}) = 0$$

since the non-zero relative virtual displacement are not compatible with the constraints. Hence it can be deduced that *the linking forces do not contribute to the generalized forces acting on the global system.*

Their absence from the evaluation of the generalized forces is one of the attractive aspects of Lagrangian mechanics.

b. Elastic forces

An elastic body can be defined as a body for which any produced work is stored in a recoverable form, thus giving rise to a variation of internal energy

$$\delta E_{p\text{int}} = \sum_{i=1}^3 \sum_{k=1}^N \frac{\partial E_p}{\partial u_{ik}} \delta u_{ik} = - \sum_{s=1}^n Q_s \delta q_s$$

with the generalized forces of elastic origin

$$Q_s = - \frac{\partial E_{p\text{int}}}{\partial q_s} \quad (2.5.3)$$

c. Dissipative forces

The dissipative force may be characterised by the fact that it remain parallel and in opposite direction to the velocity vector and is a function of its modulus. Therefore a dissipative force acting of a mass particle k may be expressed in the form

$$\mathbf{X}_k = -C_k f_k(v_k) \frac{\mathbf{v}_k}{v_k}$$

or in terms of components

$$X_{ik} = -C_{ik} f_k(v_k) \frac{v_{ik}}{v_k} \quad (2.5.4)$$

where is

C_k is a constant

$f_k(v_k)$ is the function expressing velocity dependence

v_k is the absolute velocity of particle k :

$$v_k = |\mathbf{v}_k| = \sqrt{\sum_{i=1}^3 v_{ik}^2} = \sqrt{\sum_{i=1}^3 \dot{u}_{ik}^2}$$

The virtual work of the dissipative forces acting on the system is

$$\sum_{s=1}^n Q_s \delta q_s = \sum_{i=1}^3 \sum_{k=1}^N X_{ik} \delta u_{ik} = \sum_{i=1}^3 \sum_{k=1}^N \sum_{s=1}^n X_{ik} \frac{\partial u_{ik}}{\partial q_s} \delta q_s$$

From here

$$Q_s = -\sum_{i=1}^3 \sum_{k=1}^N C_k f_k(v_k) \frac{v_{ik}}{v_k} \frac{\partial u_{ik}}{\partial q_s} \quad (2.5.5)$$

By noticing that

$$v_{ik} = \frac{du_{ik}}{dt} = \frac{\partial u_{ik}}{\partial t} + \sum_{r=1}^n \frac{\partial u_{ik}}{\partial q_r} \delta q_r^i$$

$$\frac{\partial v_{ik}}{\partial \dot{q}_s} = \frac{\partial u_{ik}}{\partial q_s}$$

it is possible to write

$$Q_s = -\sum_{i=1}^3 \sum_{k=1}^N C_k f_k(v_k) \frac{v_{ik}}{v_k} \frac{\partial v_{ik}}{\partial \dot{q}_s} = -\sum_{k=1}^N C_k f_k(v_k) \frac{\partial}{\partial \dot{q}_s} \left[\frac{1}{2} \sum_{i=1}^3 v_{ik}^2 \right] =$$

$$= -\sum_{k=1}^N C_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_s} \quad (2.5.6)$$

Let us introduce the *dissipative function* D as

$$D = \sum_{k=1}^N \int_0^{v_k} C_k f_k(v) dv \quad (2.5.7)$$

and thus

$$Q_s = -\frac{\partial D}{\partial \dot{q}_s} \quad (2.5.8)$$

By assuming that the dissipative function D is homogeneous of order h in the generalized velocities on gets

$$\frac{d}{dt}(E_k + E_p) = -hD$$

The order h of the dissipation function is

$h = 1$ dry friction

$h = 2$ viscous damping

$h = 3$ aerodynamic drag

External conservative forces

When the external forces are conservative, their virtual work remain zero during a cycle

$$\delta A = \int Q_s \delta q_s = 0$$

and a potential energy is possible to use for definition of generalized force

$$Q = -\frac{\partial E_{pext}}{\partial q_s} \quad (2.5.9)$$

External non-conservative forces

If the external forces are of the non-conservative type, the generalized force is

$$Q_s \sum_{i=1}^3 \sum_{k=1}^N X_{ik} \frac{\partial u_{ik}}{\partial q_s} \quad (2.5.10)$$

Lagrange equation of motion in the general case of non-conservative systems with rheonomic constraints may be explicitly expressed in the form

$$\boxed{\frac{d}{dt} \left(\frac{\partial E_k}{\partial \dot{q}_s} \right) - \frac{\partial E_k}{\partial q_s} + \frac{\partial E_p}{\partial q_s} + \frac{\partial D}{\partial \dot{q}_s} = Q_s(t) \quad s = 1, 2, \dots, n} \quad (2.5.11)$$

3. VIBRATIONS

We owe to Lord Rayleigh the formulation of the principles relative to theory of vibration such as they are applied and taught nowadays. In his remarkable treatise entitled *Theory of sound* and published in 1887 he introduced concept of oscillations of a linear system and showed the existence of natural modes and natural frequencies for discrete as well as continuous systems. His work remains valuable in many ways even though he was concerned with acoustics rather than with structural mechanics.

Vibration is in general a motion periodic in time and is used to describe oscillation in mechanical systems. In most cases, the general purpose is to prevent or attenuate the vibrations, because of their detrimental effects, such as fatigue failure of components and generation of noise. However, there are some applications where vibrations are desirable and are usefully employed, as in vibration conveyers, vibrating sieves, etc.

Because of their constant aim to minimize the stress in structures, the designers were the first who needed to get vibration and structural dynamic under control. During the next years, they had to limit the scope of their analysis and apply methods that could be handled by the available computational means.

Vibrations may be classified into three categories:

- *Free vibrations* can occur only in conservative systems where there is no friction, damping and exciting force. Here, the total mechanical energy, which is due to the initial conditions, is conserved and exchange can take place between the kinetic and potential energies.
- *External forces that excite the system cause forced vibrations*. The exciting forces supplies energy continuously to compensate for that dissipated by damping.
- *Self-excited vibrations* are periodic oscillations of the limit cycle type and are caused by some nonlinear phenomenon. The energy required to maintain the vibrations is obtained from a non-alternating power source. In this case, the vibrations themselves create the periodic force.

3.1 Single-degree-of-freedom systems

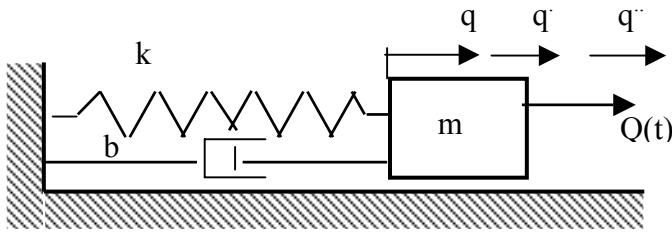


Fig. 3.1

Let us consider the model shown in fig.3.1. Displacement q is measured from the stable equilibrium position of the system, the velocity \dot{q} , the acceleration \ddot{q} is measured positive in the

positive direction of displacement. The equation of motion is:

$$m\ddot{q} + b\dot{q} + kq = Q(t) \quad (3.1.1)$$

where it is: k stiffness constant

b damping constant

This system is one of the simplest dynamic systems in which elastic, dissipating and inertia forces interact. In torsion system the mass m will be replaced by mass moment of inertia I and the force $Q(t)$ by the moment $M(t)$. The solution of the differential equation of motion is composed of two parts: the solution of the homogenous equation

$$m\ddot{q} + b\dot{q} + kq = 0$$

When we introduce

$$\text{the damping factor } \delta = \frac{b}{2m} \left[\frac{\text{rad}}{\text{s}} \right]$$

$$\text{and natural circular frequency of non-damped system } \Omega_0 = \sqrt{\frac{k}{m}} \left[\frac{\text{rad}}{\text{s}} \right]$$

we obtain

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = 0 \quad (3.1.2)$$

The solution of this homogenous equation is

$$q_h = e^{-\delta t} (A \cos \Omega t + B \sin \Omega t) = C e^{-\delta t} \sin(\Omega t - \varphi) \quad (3.1.3)$$

A, B or C, φ are the integration constants, which can be determined from the initial conditions.

The circular frequency of damped system is

$$\Omega = \sqrt{(\Omega_0^2 - \delta^2)} \left[\frac{\text{rad}}{\text{s}} \right]$$

The particular solution will be derived from the equation

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = \frac{Q(t)}{m} \quad (3.1.4)$$

The solution depends on the form of the force $Q(t)$.

3.1.1 The force of excitation is harmonic

Very important case for practical applications is when the applied force is harmonic represented in complex notation:

$$\tilde{Q}(t) = \tilde{Q}_0 e^{i\omega t} = Q_0 e^{i\varphi_F} e^{i\omega t} = Q_0 e^{i(\omega t + \varphi_F)} \quad (3.1.5)$$

Using the properties of complex numbers we write the equation (3.1.1) in the form:

$$\left(b + i\omega m + \frac{k}{i\omega} \right) \dot{\tilde{q}} = \tilde{Q}(t) \quad (3.1.6)$$

We define the *complex mechanical impedance* as the ratio of the force and velocity:

$$\tilde{Z} = Z e^{i\varphi_z} = \frac{Q(t)}{\dot{\tilde{q}}} = b + i \left(m\omega - \frac{k}{\omega} \right) \quad (3.1.7)$$

Z is the modulus of mechanical impedance

$$Z = \sqrt{(\text{Re}(\tilde{Z}))^2 + (\text{Im}(\tilde{Z}))^2} = \frac{k}{m} \sqrt{\left(1 - \frac{\omega^2}{\Omega_0^2} \right)^2 + \left(2b_r \frac{\omega}{\Omega_0} \right)^2} \quad (3.1.8)$$

The phase angle is defined as

$$\varphi_z = \text{arctg} \frac{m\Omega_0 \left(\frac{\omega}{\Omega_0} - \frac{\Omega_0}{\omega} \right)}{b} \quad (3.1.9)$$

The velocity is possible to express as the imaginary part of complex velocity:

$$\dot{q}_p = \text{Im}(\dot{\tilde{q}}) = \frac{Q_0}{\omega Z} \sin(\omega t + \varphi_F - \varphi_z) \quad (3.1.10)$$

The complex displacement is the derivative of the velocity...

$$\tilde{q}_p = \frac{\dot{\tilde{q}}}{i\omega} = \frac{\tilde{Q}}{i\omega\tilde{Z}} = \frac{Q_0}{i\omega Z} \exp[i(\omega t + \varphi_F - \varphi_z)] = \tilde{H}(i\omega)\tilde{Q} \quad (3.1.11)$$

$\tilde{H}(i\omega)$ is called *complex transfer function of the mechanical system*.

The displacement is the imaginary part of the of the complex solution

$$q_p = \text{Im}\{\tilde{q}_p\} = -\frac{Q_0}{\omega Z} \cos(\omega t + \varphi_F - \varphi_z) = \frac{Q_0}{\omega Z} \sin\left[\omega t + \varphi_F - \left(\varphi_z - \frac{\pi}{2} \right) \right] \quad (3.1.12)$$

In next description we notify the expression $\varphi_Z - \pi/2 = \varphi$. It is called the *phase retard*.

Using the equation (3.1.8) we get

$$q_p = s_0 \sin(\omega t + \varphi_F - \varphi) \quad (3.1.13)$$

s_0 is the amplitude of forced response:

$$s_0 = \frac{Q_0}{k \sqrt{(1 - \eta^2)^2 + (2b_r \eta)^2}} \quad (3.1.14)$$

Here is $\eta = \frac{\omega}{\Omega_0}$ the *frequency ratio* and $b_r = \frac{\delta}{\Omega_0} = \frac{b}{2m\Omega_0}$ is the *damping ratio*.

The expression $\frac{Q_0}{k} = q_{st}$ is the deformation of the spring statically loaded by the amplitude of exciting force sometimes called *static deformation*. So is defined *frequency transfer* or *the coefficient of amplification*:

$$\lambda = \frac{s_0}{q_{st}} = \frac{1}{\sqrt{(1 - \eta^2)^2 + (2b_r \eta)^2}} \quad (3.1.15)$$

This dimensionless quantity is plotted in the *amplitude diagram* shown in Fig. 3.2

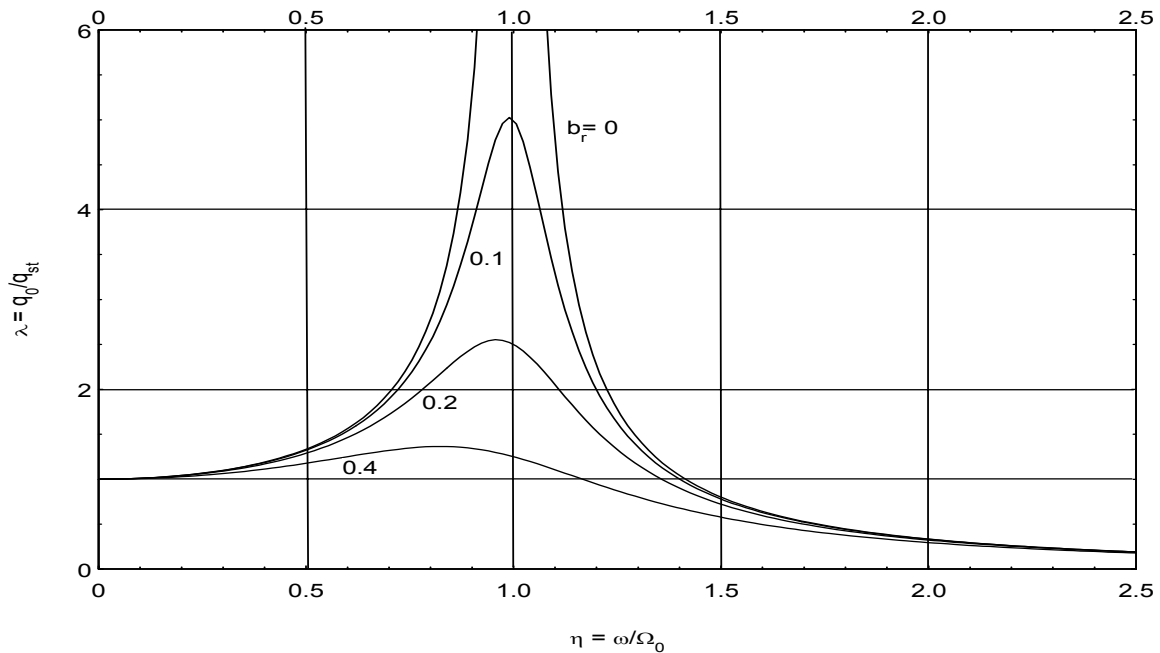


Fig. 3.2

The amplitude contains the family of curves one for each value of damping ratio b_r . All curves lie below the one for zero damping. Thus we see that the amplitude of forced vibration is diminished by damping. By zero damping and by $\eta = 1$ or $(\omega = \Omega_0)$, the amplitude goes to

infinity. This state is called *the resonance*. The maxima of various curves of damped vibrations do not occur any long for $\eta = 1$ but at a smaller value:

$$\eta_m = \sqrt{1 - 2b_r^2} < 1$$

and the maximum frequency transfer is:

$$\lambda_{\max} = \frac{1}{2b_r \sqrt{1 - b_r^2}} \quad (3.1.16)$$

When the damping ratio is very small $b_r \ll 1$ it is possible to use $\lambda_{\max} \approx \frac{1}{2b_r}$.

The phase angle is given by the expression

$$\operatorname{tg} \varphi_z = \frac{\operatorname{Im}(\tilde{Z})}{\operatorname{Re}(\tilde{Z})} = \frac{m\Omega_0 \left(\frac{\omega}{\Omega_0} - \frac{\Omega_0}{\omega} \right)}{b} \quad (3.1.17)$$

In the equation (3.1.13) we introduced the angle φ which is

$$\operatorname{tg} \varphi = \operatorname{tg} \left(\varphi_z - \frac{\pi}{2} \right) = -\frac{1}{\operatorname{tg} \varphi_z}$$

Substituting in this expression we obtain the phase

$$\varphi = \operatorname{arctg} \frac{2b_r \eta}{1 - \eta^2} \quad (3.1.18)$$

The equation (3.1.18) is possible to plot in *the phase angle diagram* (Fig. 3.3), which is also of considerable interest. For no damping, it is seen that below resonance the force and the displacement are in phase ($\varphi = 0$),

For damping different from zero the other curves represent the phase angle reach the phase of 90° by resonance. By measuring the phase angle it is possible to determine the exact point of resonance. So we have been derived the particular solution.

The general solution consists of the damped free vibration superposed on the forced vibration.

$$q = Ce^{-\delta t} \sin(\Omega t + \varphi_0) + s_0 \sin(\omega t + \varphi_F - \varphi_Z) \quad (3.1.19)$$

After a short time the damped free vibration disappears and the forced vibration alone persists. Therefore the forced vibration is also called the *sustained vibration*, while the free vibration is known as the *transient vibration*.

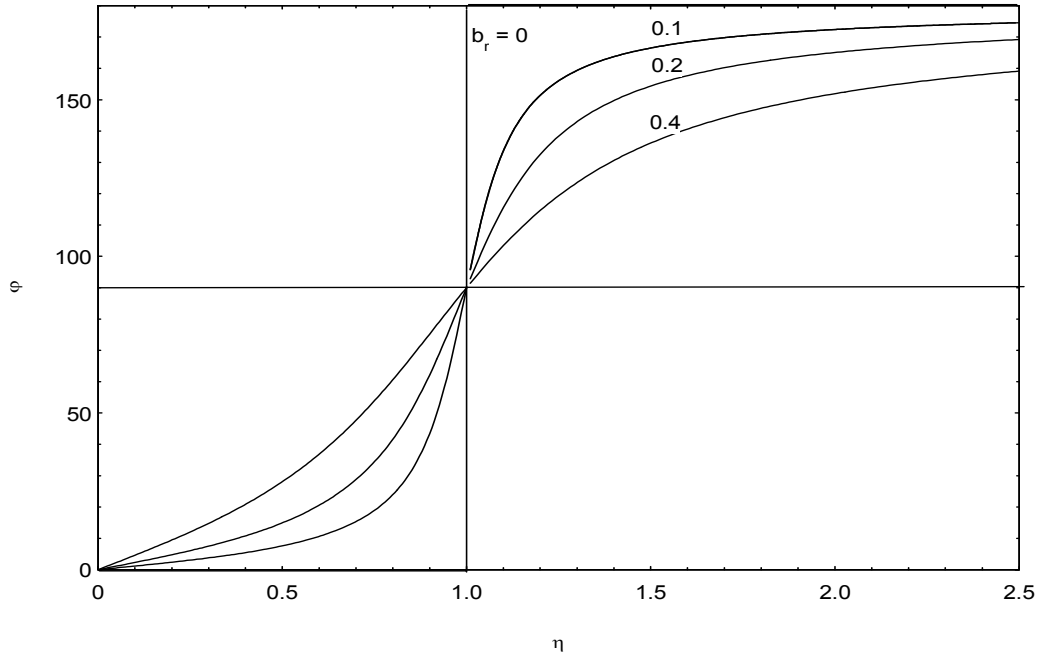


Fig. 3.3

3.1.2 The excitation by rotating mass

Very often the excitation is caused by unbalanced rotating mass. Such case is shown on Fig. 3.4. The unbalanced rotor is represented by the mass m_1 placed on the eccentricity e from the axes of rotation. m is the total mass of the equipment. The resultant stiffness is k and the damping is b . The vertical inertia force is

$$F = m_1 e \omega^2 \sin \omega t$$

The equation of motion is

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = \frac{m_1}{m} e \omega^2 \sin \omega t$$

When we compare this equation with (3.1.4) we see that both equation

$$Q(t) = m_1 e \omega^2 \sin \omega t$$

Therefore the solution is identical to that excited by harmonic force. The amplitude of sustained vibrations is given by following formula

$$s_0 = \frac{m_1 e \eta^2}{m \sqrt{(1 - \eta^2)^2 + (2b_r \eta)^2}} \quad (3.1.20)$$

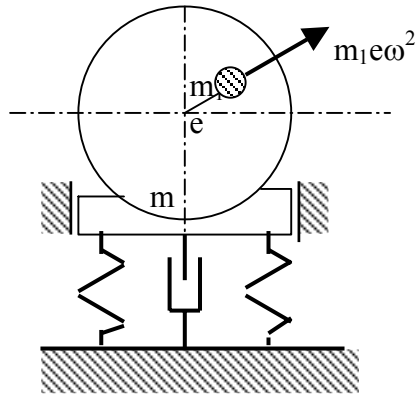


Fig. 3.4

The frequency transfer will be

$$\lambda = \frac{s_0}{\frac{m_1 e}{m}} = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + (2b_r \eta)^2}} \quad (3.1.21)$$

This expression is possible to plot in the amplitude diagram (Fig. 3.5).

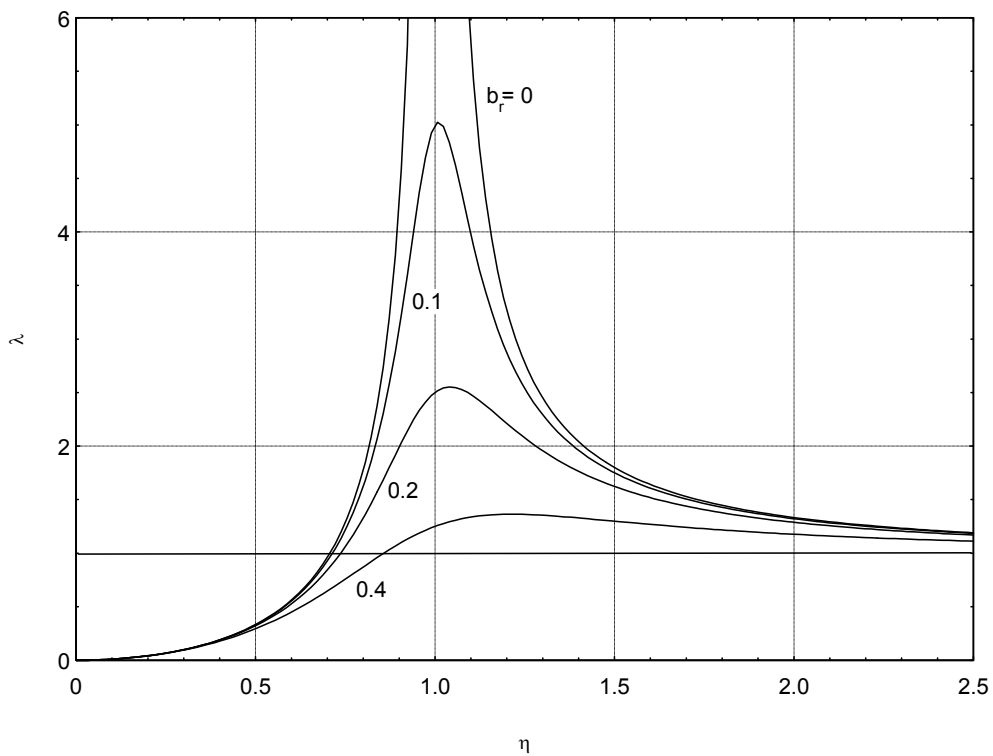


Fig. 3.5

3.1.3 The force is general function of time

Very often the excitation force is a general function of time. The particular solution is given by *Duhamel integral*:

$$q_p = \frac{1}{m\Omega_0} \int_0^t Q(\tau) e^{-\delta(t-\tau)} \sin \Omega(t-\tau) d\tau \quad (3.1.22)$$

The analytical solution of this integral is possible for simple functions of general force. But the Duhamel integral is possible advantageous to solve numerically, even when the force is obtained a table. In Tab. 3.1 the algorithm for solution on PC is shown. We suppose that the force is given tabular. The numerical integration is performed by using the *Simpson rule*

$$\int_{x=x_n}^{x_{n+2}} f(x) dx \approx \frac{h}{3} (y_n + 4y_{n+1} + y_{n+2})$$

The length h of integration interval is constant and the number N of intervals must be even. Algorithm of numerical solution of response by using Duhamel integral is shown in Tab. 3.1.

3.1.4 Exciting force is a periodic function of time

In many cases the exciting force is a periodic function of time. It means that its value repeat after the period T_F

$$F(t) = F(t + T_F) = F(t + nT_F) \quad \text{for } n = 1, 2, \dots, n$$

In such case it is possible expand the force into Fourier series

$$F(t) = \sum_{i=0}^{\infty} (F_{1i} \cos i\omega t + F_{2i} \sin i\omega t) \quad (3.1.23)$$

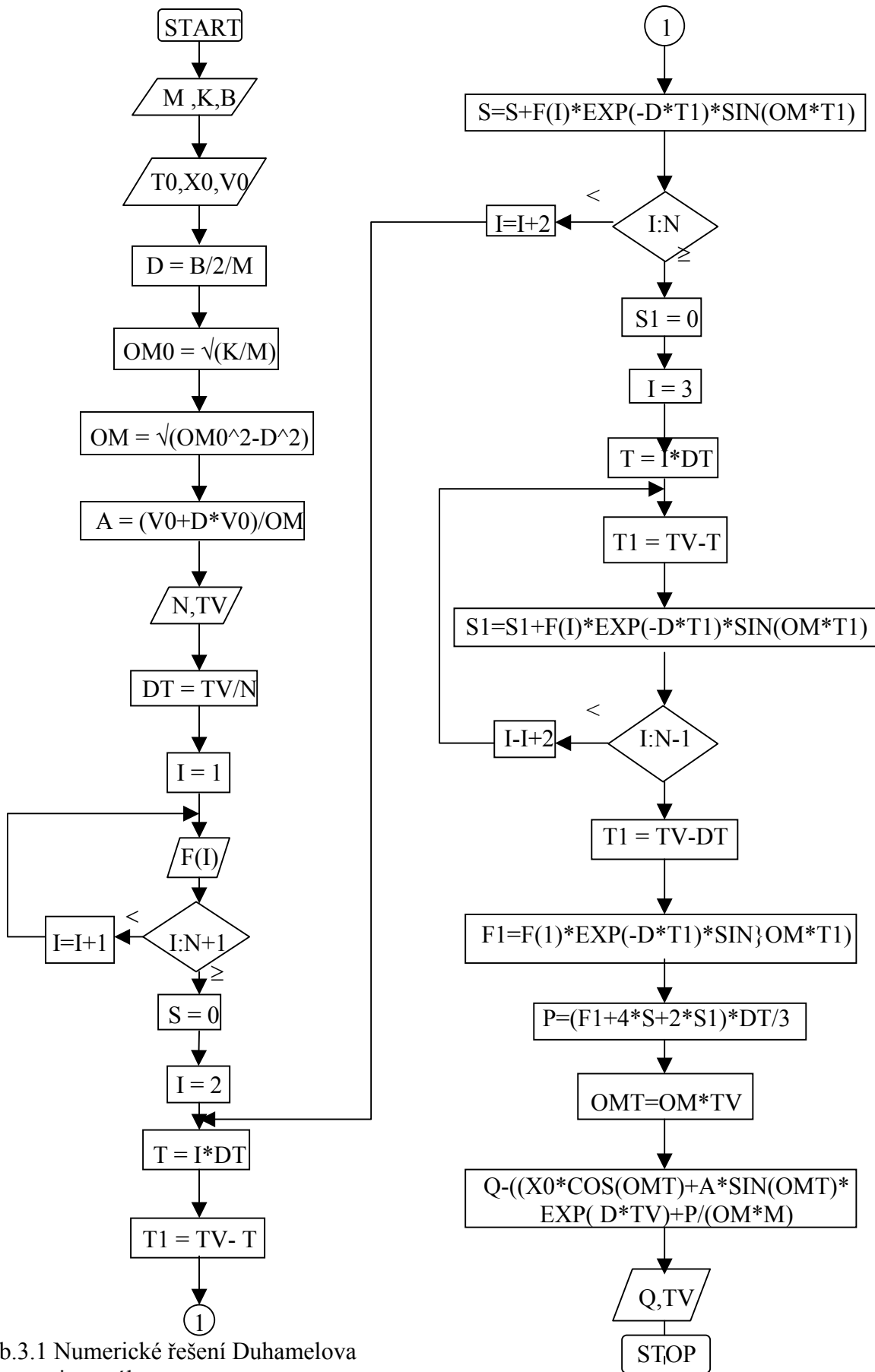
where $\omega = 2\pi/T_F$

The determination of Fourier coefficients is well known from mathematics:

$$\begin{aligned} F_{10} &= \frac{1}{T_F} \int_0^{T_F} F(t) dt \\ F_{1i} &= \frac{2}{T_F} \int_0^{T_F} F(t) \cos(i\omega t) dt \\ F_{2i} &= \frac{2}{T_F} \int_0^{T_F} F(t) \sin(i\omega t) dt \end{aligned} \quad (3.1.24)$$

The equation (3.1.4) by using (3.1.23) will have the form

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = \frac{1}{m} \sum_{i=0}^n (F_{1i} \cos(i\omega t) + F_{2i} \sin(i\omega t)) \quad (3.1.25)$$



Tab.3.1 Numerické řešení Duhamelova integrálu

In practical applications we do't take infinity number of Fourier coefficients, but only n .

The right hand side of (3.1.25) we arrange when used

$$F_{1i} = F_i \sin \varphi_{Fi} \quad F_{2i} = F_i \cos \varphi_{Fi} \quad \text{for } i = 1, 2, \dots$$

So it is

$$F_i = \sqrt{F_{1i}^2 + F_{2i}^2} \quad \varphi_{Fi} = \arctg \frac{F_{1i}}{F_{2i}} \quad (3.1.26)$$

Now we can re-write the equation (3.1.25) in the form

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = \frac{F_{10}}{m} + \frac{1}{m} \sum_{i=1}^n F_i \sin(i\omega t + \varphi_{Fi}) \quad (3.1.27)$$

If holds the *law of superposition* we can determine the response for each component of the force separately and then the resultant response is given by adding all particular calculated responses due to separate harmonic terms of (3.1.27).

The general solution is obtained again from the homogeneous and particular solutions.

$$q = q_h + q_0 + \sum_1^n q_{pi} \quad (3.1.28)$$

In this equation is

$$q_0 \frac{F_{10}}{k} = \frac{F_{10}}{m\Omega_0^2} \quad (3.1.29)$$

$$q_h = C e^{-\delta t} \sin(\Omega + \varphi_0) \quad (3.1.30)$$

$$q_{pi} = s_{0i} \sin(i\omega t + \varphi_{Fi} - \varphi_i) \quad (3.1.31)$$

The amplitude of particular solution is done by

$$s_{0i} = \frac{F_i}{k \sqrt{(1 - i\eta^2)^2 + (2b_r i\eta)^2}} \quad (3.1.32)$$

and

$$\varphi_i = \arctg \frac{2b_r i\eta}{1 - (i\eta)^2} \quad (3.1.33)$$

From (3.1.32) we see that, that particular harmonic components magnified the response according the value of F_i and the order i .

Very often the course of forces is known from measurements. In such case the components of Fourier series is also possible to get from measured values. We consider the period of the force is T_F and the number of measurements is $N+1$. The time interval will be $\Delta t = T_F/N$, and the time from the beginning of force action is $t_j = j\Delta t$.

We introduce the value

$$\omega t_j = \frac{2\pi}{T_F} j \frac{T_F}{N} = \frac{2j\pi}{N}$$

The measured function will be denoted by $Y(t_j) = Y_j$. The coefficients of Fourier series are determined by

$$F_{10} = \frac{1}{N} \sum_{j=0}^N Y_j$$

$$F_{1i} = \frac{2}{N} \sum_{j=0}^N Y_j \cos\left(\frac{2ij\pi}{N}\right) \quad \text{for } i = 1, 2, \dots, l$$

$$F_{2i} = \frac{2}{N \sum_{j=0}^N Y_j \sin\left(\frac{2ij\pi}{N}\right)}$$

The numerical solution on PC is without any problem. The algorithm is shown in Tab. 3.2.

3.1.5 The kinematical excitation

The exciting, considered so far has been done by the force acting on the moving mass. Now we shall consider that the frame move harmonically according the formula

$$q_z(t) = h \sin \omega t \quad (3.1.34)$$

Such case is sometimes called *seismic excitation*.

The differential equation of motion of the moving mass will be

$$-b[\dot{q} - \dot{q}_z(t)] - k[q - q_z(t)] = m\ddot{q} \quad (3.1.35)$$

After arrangement of (3.1.35) we get

$$m\ddot{q} + b\dot{q} + kq = b\dot{q}_z(t) + kq_z(t) = f(t) \quad (3.1.36)$$

It is seen that the motion is harmonic. If we consider that the base move according (3.1.34) the function $f(t)$ is

$$f(t) = bh\omega \cos \omega t + kh \sin \omega t$$

Using the notation $\Omega_0 = \sqrt{\frac{k}{m}}$ and $\delta = \frac{b}{2m}$ the equation (3.1.36) obtains the form

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = 2\delta h\omega \cos \omega t + \Omega_0^2 h \sin \omega t \quad (3.1.37)$$

The right hand side of (3.1.37) is possible to simplify by notation

$$2\delta h\omega = p_0 \sin \varphi_z$$

$$\Omega_0^2 h = p_0 \cos \varphi_z$$

From here we get

$$p_0 = \sqrt{\Omega_0^4 h^2 + (2\delta h\omega)^2} = \Omega_0^2 h \sqrt{1 + (2b_r \eta)^2}$$

$$\varphi_z = \arctg(2b_r \eta)$$

By using of these expressions the equation (3.1.37) obtains the form

$$\ddot{q} + 2\delta \dot{q} + \Omega_0^2 q = p_0 \sin(\omega t + \varphi_z) \quad (3.1.38)$$

The particular solution of this equation will be

$$q_p = s_0 \sin(\omega t + \varphi_z - \varphi) \quad (3.1.39)$$

with the amplitude of harmonic motion of the mass

$$s_0 = \frac{h \sqrt{1 + (2b_r \eta)^2}}{\sqrt{(1 - \eta^2)^2 + (2b_r \eta)^2}} \quad \text{or} \quad \lambda = \frac{s_0}{h} = \frac{\sqrt{1 + (2b_r \eta)^2}}{\sqrt{(1 - \eta^2)^2 - (2b_r \eta)^2}} \quad (3.1.40)$$

The course of frequency transfer λ is shown in Fig. 3.6.

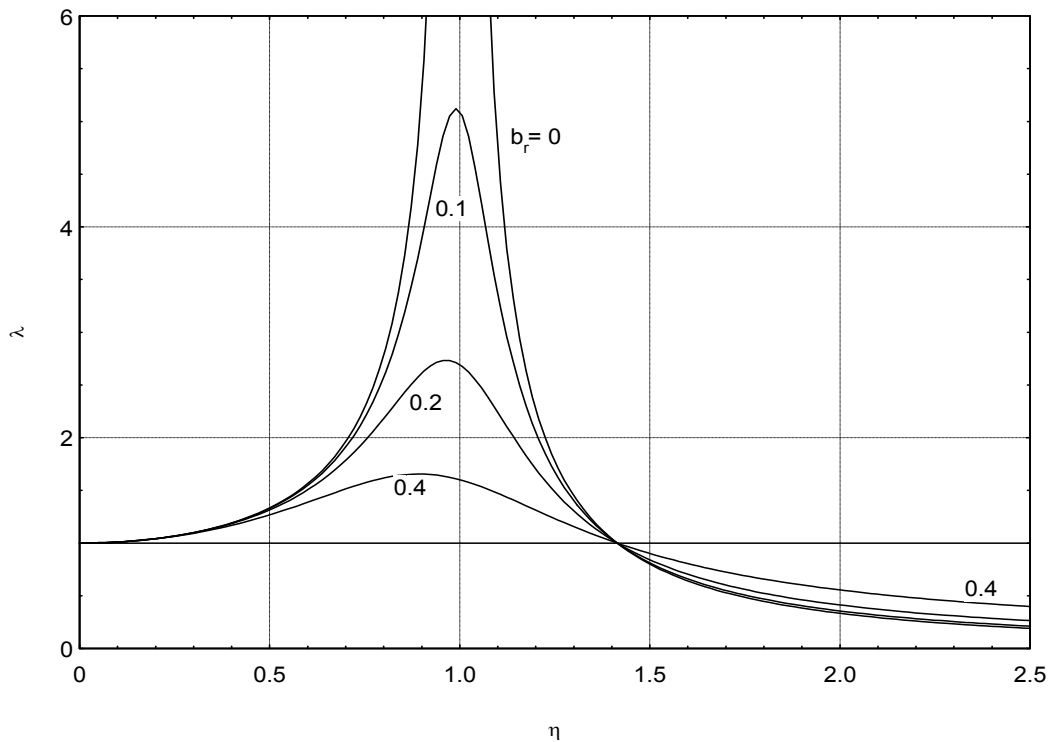


Fig. 3.6

The phase is given by the formula

$$\varphi = \text{arctg} \frac{2b_r \eta^3}{1 - \eta^2 + (2b_r \eta)^2} \quad (3.1.41)$$

The phase diagram is on Fig. 3.7.

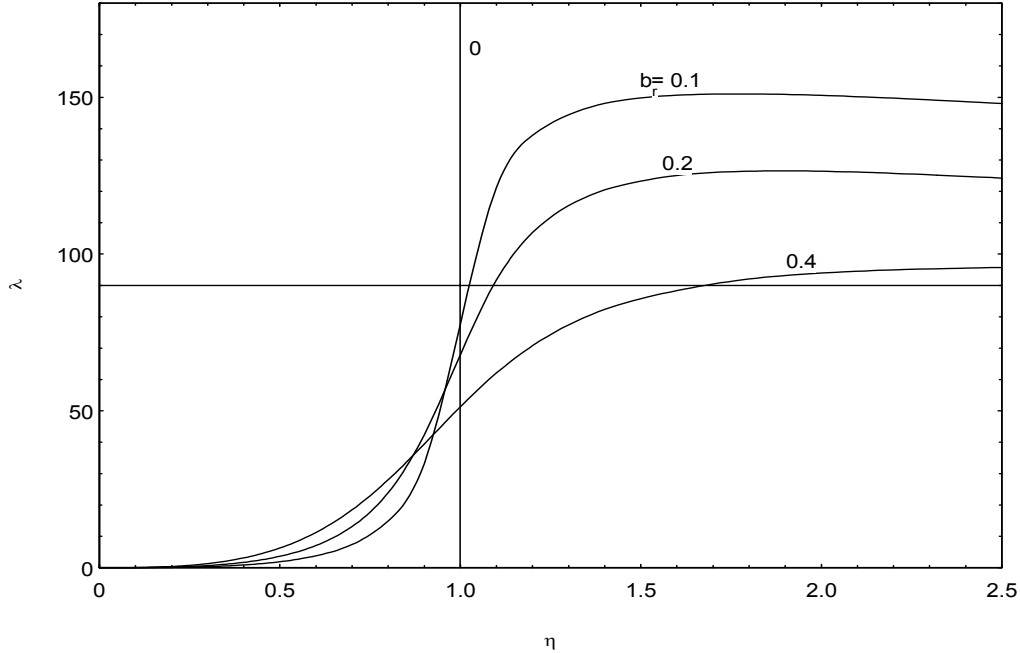


Fig. 3.7

3.1.6 Theory of vibration isolation

An unbalanced machine has to be installed in a structure where vibration is undesirable. Such situation is not uncommon. An elevator motor in a building and the machine in an automobile are examples. The problem consists in mounting the machine in such a manner that no vibration will appear in the structure to which it is attached. We consider the rigid frame. For this case it is possible to use the Fig. 3.1. The force is transmitted from the moving mass to the frame through the spring and the damper.

$$F_T = kq + b\dot{q}$$

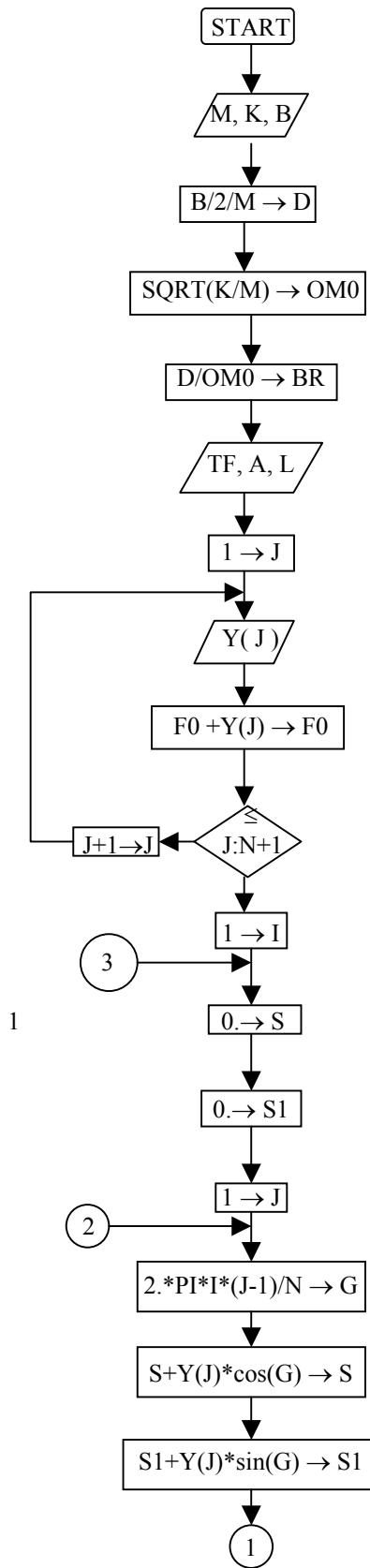
The vibrating mass moves according the law

$$q = s_0 \sin(\omega t + \varphi_F - \varphi) \quad \dot{q} = s_0 \omega \cos(\omega t + \varphi_F - \varphi)$$

Therefore we get for the transmitted force

$$F_T = s_0 k \sin(\omega t + \varphi_F - \varphi) + s_0 b \omega \cos(\omega t + \varphi_F - \varphi) \quad (3.1.41)$$

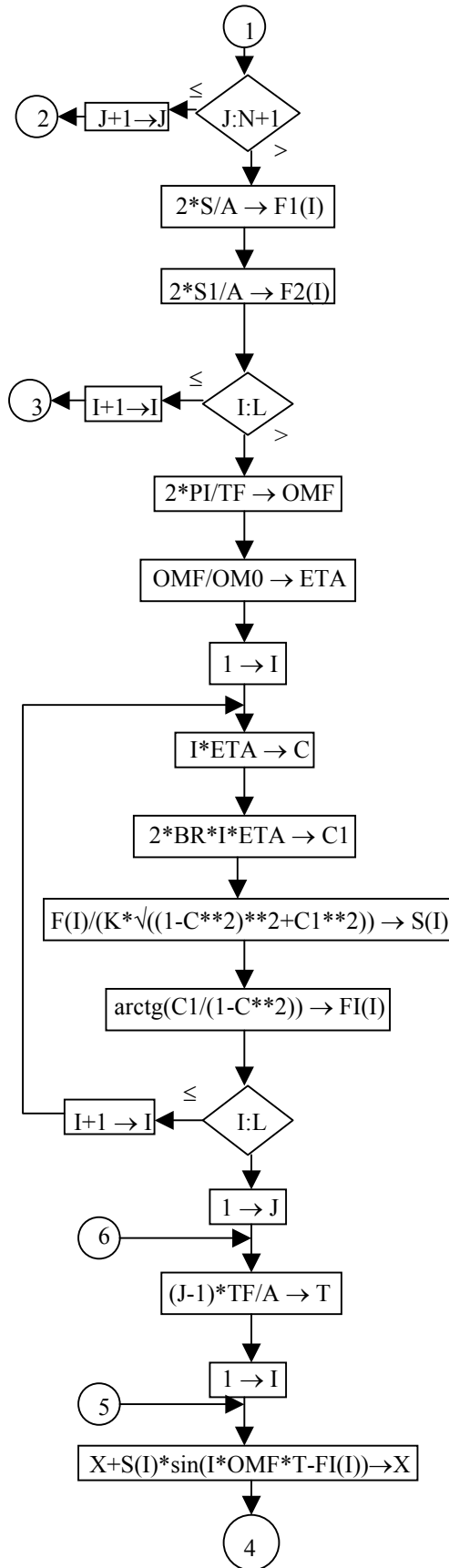
The equation (3.1.41) is better transform in the form

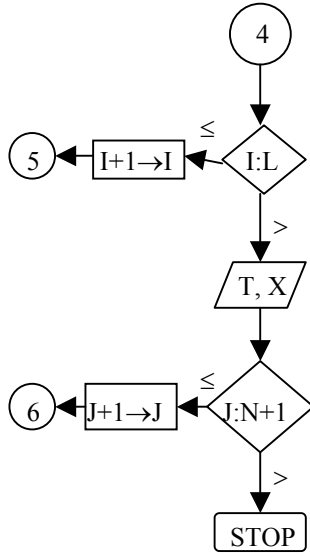


1

4

5 I+1→I ≤ I:L





Tab. 3.2 . The algorithm of response on periodic force.

$$F_T = F_{T0} \sin(\omega t + \varphi_F - \varphi + \varphi_T) \quad (3.1.42)$$

This form we get by putting

$$s_0 k = F_{T0} \cos \varphi_T$$

$$s_0 b \omega = F_{T0} \sin \varphi_T$$

From these equations we get

$$F_{T0} = s_0 k \sqrt{1 + (2b_r \eta)^2} \quad (3.1.43)$$

Substituting for s_0 from equation (3.1.14) we obtain the amplitude of *transmitted force*

$$F_{T0} = \frac{Q_0 \sqrt{1 + (2b_r \eta)^2}}{\sqrt{(1 - \eta^2)^2 + (2b_r \eta)^2}} \quad (3.1.43)$$

We can use the *transmissibility factor* $\lambda = \frac{\text{transmitted force}}{\text{impressed force}} = \frac{F_{T0}}{Q_0}$ and get

$$\lambda = \frac{\sqrt{1 + (2b_r \eta)^2}}{\sqrt{(1 - \eta^2)^2 + (2b_r \eta)^2}} \quad (3.1.44)$$

The form of equation (3.1.44) is the same as (3.1.41). Therefore the amplitude diagram is the same as in Fig. 3.6.

The phase is given by the formula

$$\varphi_T = \text{arctg}(2b_r \eta) \quad (3.1.45)$$

3.2 Vibrations of n-degree-of-freedom systems

Exactly all systems have the mass as well as stiffness distributed continuously in the whole system. Even in the simplest system – the mass point on a spring – the mass is not concentrated in the point and the spring is not mass less. When we need only one natural frequency we construct a mechanical model like in the Fig.1. This model represents the searched properties with enough accuracy. Simpler model enables more easy mathematical calculation and many times gives a sufficient accuracy. If we need to know also the higher

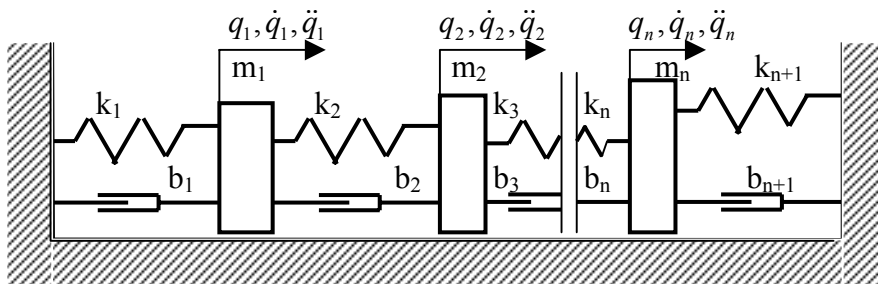


Fig. 3.8

natural frequencies, the model must be more complicated. Usually we design more complicated mechanical model – so called *linear discrete model*. We obtain as many natural frequencies, and as many natural modes as they are degrees of freedom.,

Let we consider the model on the Fig. 3.8. The equations of motion is possible to obtain by using of Lagrange equations, or Hamilton's principle. In such simple models it is possible to write directly for each released mass.

$$\begin{aligned}
 m_1 \ddot{q}_1 + (b_1 + b_2) \dot{q}_1 - b_2 \dot{q}_2 + (k_1 + k_2) q_1 - k_2 q_2 &= Q_1(t) \\
 m_2 \ddot{q}_2 - b_2 \dot{q}_1 + (b_2 + b_3) \dot{q}_2 - b_3 \dot{q}_3 - k_2 q_1 + (k_2 + k_3) q_2 - k_3 q_3 &= Q_2(t) \\
 \bullet \\
 \bullet \\
 m_n \ddot{q}_n - b_n \dot{q}_{n-1} + (b_n + b_{n+1}) \dot{q}_n - k_n q_{n-1} + (k_n + k_{n+1}) q_n &= Q_n(t)
 \end{aligned}$$

We obtained n simultaneous differential equations of second order with constant coefficients. When the number of degrees increases the solution is difficult and not providing an easy survey. Therefore we write the set of equation of motion in matrix notation:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}(t) \tag{3.2.1}$$

In this equation is:

$\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ displacement, velocity, acceleration, respectively. They are expressed by a column matrix $\mathbf{q}^T = [q_1, q_2, \dots, q_n]$. $\mathbf{Q}^T(t) = [Q_1, Q_2, \dots, Q_n]$ is the vector of time depending exciting forces. \mathbf{M} is the mass matrix, \mathbf{B} is the matrix of damping and \mathbf{K} is the stiffness matrix. At conservative systems are squared, symmetric and are of order n . When we consider model on Fig. 3. 8, they have the form:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & m_n \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_1 + b_2 & -b_2 & 0 & 0 \\ -b_2 & b_2 + b_3 & -b_3 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -b_n & b_n + b_{n+1} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -k_n & k_n + k_{n+1} \end{bmatrix}$$

3.2.1 Free, un-damped vibrations

Free, un-damped vibrations described by equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \quad (3.2.2)$$

is important for the next solutions. Therefore we concern on it in detail.

We consider the solution of (3.2.2)

$$\mathbf{q} = \mathbf{u}e^{i\Omega t}$$

Here \mathbf{u} is a vector of amplitudes of harmonic motion $\mathbf{u}^T = [u_1, u_2, \dots, u_n]$. Ω is the circular frequency. Equation (3.2.2) by using the assumption of harmonic motion will have the form

$$(\mathbf{K} - \Omega^2\mathbf{M})\mathbf{u} = \mathbf{0} \quad (3.2.3)$$

(3.2.3) represents the set of homogenous equations. For non-trivial solution must be the determinant equal to zero

$$\det|\mathbf{K} - \Omega_0^2\mathbf{M}| = 0 \quad (3.2.4)$$

This determinant is called the *frequency determinant*. When we developed this determinant we get the *frequency equation* of n order for Ω_0^2 :

$$a_n\Omega_0^{2n} + a_{n-1}\Omega_0^{2(n-1)} + \dots + a_1\Omega_0^2 + a_0 = 0$$

While the matrices are positive and definite the roots of this equations are real values:

$$0 \leq \Omega_{01} \leq \Omega_{02} \leq \dots \leq \Omega_{0n}$$

When we substitute a natural frequency in (3.2.3) we obtain again the set of homogenous equations. Therefore it is necessary to divide each equation by one element of the amplitude vector u_{ri} . We get for example

$$\mathbf{v}_r^T = \left[\frac{u_{r1}}{u_{r1}}, \frac{u_{r2}}{u_{r1}}, \dots, \frac{u_{rn}}{u_{r1}} \right]$$

By this way it is possible to create n variable sequences. The vectors \mathbf{v}_r gives the shape of the vibrating system but not the absolute value of the displacements of its members.

Therefore these vectors are called modal vectors. From n sequences we choose that one, whose maximum absolute value is 1. This process is called *normalization*. The normalization is possible to carry out by using one of the following procedure...

$$\mathbf{v}_r^T \mathbf{v}_r = 1$$

$$\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r = 1$$

$$\mathbf{v}_r^T \mathbf{K} \mathbf{v}_r = 1$$

Which procedure is advantageous we shall see later.

The displacements that belong to r mode are given by the following equation

$$\tilde{\mathbf{q}}_r = \mathbf{v}_r e^{i\Omega_{0r}t} \quad (3.2.5)$$

or in the real region

$$\mathbf{q} = \mathbf{v}_r \sin(\Omega_{0r}t + \varphi_r) \quad (3.2.6)$$

From this equation it is seen, that the mode does not change during the vibration.

The general solution of (3.2.2) is given by linear combination of all modes

$$\tilde{\mathbf{q}} = \sum_{r=1}^n \tilde{C}_r \mathbf{v}_r e^{i\Omega_{0r}t} \quad (3.2.7)$$

\tilde{C}_r are complex integration constants. In real region (3.2.7) obtains the form

$$\mathbf{q} = \sum_{r=1}^n C_r \mathbf{v}_r \sin(\Omega_{0r}t + \varphi_r) \quad (3.2.8)$$

or

$$\mathbf{q} = \sum_{r=1}^n \mathbf{v}_r (A_r \cos \Omega_{0r}t + B_r \sin \Omega_{0r}t) \quad (3.2.9)$$

The integration constants C_r , φ_r or A_r , B_r pro $r = 1, 2, \dots, n$ are determined from initial conditions. The modal vectors is possible arrange in *modal matrix*

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n] = \begin{bmatrix} v_{11} & v_{12} & \cdot & \cdot & v_{1n} \\ v_{21} & v_{22} & \cdot & \cdot & v_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{n1} & v_{n2} & \cdot & \cdot & v_{nn} \end{bmatrix} \quad (3.2.10)$$

And the natural circular frequencies in *spectral matrix*

$$\mathbf{\Omega}_0^2 = \begin{bmatrix} \Omega_{01}^2 & 0 & \cdot & \cdot & 0 \\ 0 & \Omega_{02}^2 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \Omega_{0n}^2 \end{bmatrix} \quad (3.2.11)$$

3.2.1.1 Orthogonality of vibration modes

Let we consider that researched mechanical system has natural circular frequencies $\Omega_{0r} \neq \Omega_{0s}$

Equations (3.2.3) are written in the form

$$\begin{aligned} (\mathbf{K} - \Omega_{0r}^2 \mathbf{M}) \mathbf{v}_r &= \mathbf{0} \\ (\mathbf{K} - \Omega_{0s}^2 \mathbf{M}) \mathbf{v}_s &= \mathbf{0} \end{aligned}$$

We multiply the first equation by the vector \mathbf{v}_s^T and the second one by \mathbf{v}_r^T :

$$\begin{aligned} \mathbf{v}_s^T (\mathbf{K} - \Omega_{0r}^2 \mathbf{M}) \mathbf{v}_r &= \mathbf{0} \\ \mathbf{v}_r^T (\mathbf{K} - \Omega_{0s}^2 \mathbf{M}) \mathbf{v}_s &= \mathbf{0} \end{aligned}$$

The second of these equations will be transposed

$$\mathbf{v}_s^T (\mathbf{K} - \Omega_{0s}^2 \mathbf{M}) \mathbf{v}_r = \mathbf{0}$$

Now we subtract this equation from the first one:

$$(\Omega_{0r}^2 - \Omega_{0s}^2) \mathbf{v}_s^T \mathbf{M} \mathbf{v}_r = 0$$

Because it has been supposed that $\Omega_{0r} \neq \Omega_{0s}$ must be valid

$$\mathbf{v}_s^T \mathbf{M} \mathbf{v}_r = 0 \quad (3.2.12a)$$

And by similar procedure we get

$$\mathbf{v}_s^T \mathbf{M} \mathbf{v}_r = 0 \quad (3.2.12b)$$

Always when $r \neq s$

(3.2.12a,b) are the *orthogonality relationships* between natural modes of distinct natural frequencies. In vibration of mechanical systems they say that if the system vibrate by one natural frequency it is in the system only the mode belonging to the natural this frequency. It is also possible to say: *The mode vectors belonging to various natural frequencies are orthogonal with respect to the mass matrix as well the stiffness matrix.* The quadratic forms

$$\begin{aligned}\mathbf{q}_r^T \mathbf{K} \mathbf{q}_r &= k_{yr} \\ \mathbf{q}_r^T \mathbf{M} \mathbf{q}_r &= m_{yr}\end{aligned}\quad (3.2.13)$$

are respectively called *generalized stiffness* and *generalized mass of mode r*. The orthogonality relationships is possible to write in more complex form

$$\mathbf{V}^T \mathbf{M} \mathbf{V} = \left[\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r \right] = \left[m_{yr} \right] = \mathbf{M}_y \quad (3.2.14)$$

$$\mathbf{V}^T \mathbf{K} \mathbf{V} = \left[\mathbf{v}_r^T \mathbf{K} \mathbf{v}_r \right] = \left[k_{yr} \right] = \mathbf{K}_y$$

\mathbf{V} is called the *modal matrix*. The matrices \mathbf{M}_y , and \mathbf{K}_y are diagonal. We notice that the mass matrix is *positive definite*. Therefore all generalized masses are positive.

The modal matrix is possible to use to define the *main or normal coordinates*. The normal coordinates \mathbf{y} we obtain by modal transformation:

$$\mathbf{y} = \mathbf{V}^{-1} \mathbf{q} \quad \text{or} \quad \mathbf{q} = \mathbf{V} \mathbf{y} \quad (3.2.15)$$

The solution of linear systems is very advantageous, because remove the constraints between the equations of motion.

Let we consider the un-damped system

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{Q}(t)$$

If substituting for \mathbf{q} from (3.2.15) we get

$$\mathbf{M} \mathbf{V} \ddot{\mathbf{y}} + \mathbf{K} \mathbf{V} \mathbf{y} = \mathbf{Q}(t)$$

Multiplying this equation from left by modal transformed matrix \mathbf{V}^T we obtain

$$\mathbf{M}_y \ddot{\mathbf{y}} + \mathbf{K}_y \mathbf{y} = \mathbf{V}^T \mathbf{Q}(t) = \mathbf{Q}_y(t)$$

Because the matrices \mathbf{M}_y and \mathbf{K}_y are diagonal we get n independent equations

$$m_{yr} \ddot{y}_r + k_{yr} y_r = Q_{yr}(t) \quad \text{for } r = 1, 2, \dots, n \quad (3.2.16)$$

If the modal vectors have been normalized ($\mathbf{M}_y = \mathbf{V}^T \mathbf{M} \mathbf{V} = \mathbf{E}$), the equation of motion will be

$$\ddot{\mathbf{y}} + \mathbf{K}_y \mathbf{y} = \mathbf{Q}_y(t) \quad (3.2.17)$$

and

$$\mathbf{K}_y = \mathbf{\Omega}_0^2$$

3.2.1.2 Determination of natural frequencies and modal vectors by Jacobi's method

To calculate the natural frequencies and modal vector is possible applied the method in mathematics called *eigenproblem*.

The equation (3.2.3) is possible to arrange on the form

$$(\mathbf{M}^{-1}\mathbf{K} - \Omega_{0r}^2\mathbf{E})\mathbf{u}_r = \mathbf{0}$$

We denote $\mathbf{M}^{-1}\mathbf{K} = \mathbf{A}$ and $\Omega_{0r}^2 = \lambda_r$ and substitute these values in the previous equation

$$\mathbf{A}\mathbf{u}_r = \lambda_r\mathbf{u}_r \quad (3.2.18)$$

This is the mathematical formulation of eigenproblem. If matrix \mathbf{A} is symmetrical the equation (3.2.18) is possible to solve by *Jacobi's* method. Therefore, using it to compute vibration natural frequencies of a mechanical system requires the preliminary construction of the symmetric dynamic flexibility matrix of the system.

Although it is hundred years old, Jacobi's method is still frequently used. Indeed it is characterized by an exceptional stability and a very great simplicity. It can be applied without restriction to any symmetric matrix, whether its eigenvalues are positive, negative or zero. Jacobi's algorithm consists of progressively reducing the initial symmetric matrix to the diagonal form by an infinite sequence of orthogonal transformations. To do so, we construct a series of matrices verifying the recurrence relationship

We introduce the denotation

$$\mathbf{T}_k = \mathbf{S}_k^T \mathbf{S}_{k-1}^T \dots \mathbf{S}_1^T \mathbf{A} \mathbf{S}_1 \mathbf{S}_2 \dots \mathbf{S}_k \quad \text{where } \mathbf{T}_0 = \mathbf{A}$$

The elements of matrix \mathbf{T}_k are $t_{ij}^{(k)}$ and elements of \mathbf{S}_k are $s_{ij}^{(k)}$.

We define $v_k = \sum_{i=1}^n \sum_{j=1}^n (t_{ij}^{(k)})^2$ for all cases when $i \neq j$ and $k = 0, 1, 2, \dots$

$$w_k = \sum_{j=1}^n \sum_{i=1}^n (t_{ij}^{(k)})^2 \quad \text{for } k = 0, 1, 2, \dots$$

During each of transformation steps we ask zero of not-diagonal members but so, that it will be valid

$$v_{k+1} < v_k \quad \text{a} \quad w_{k+1} = w_k \quad \text{when} \quad \lim_{k \rightarrow \infty} v_k = 0$$

During the calculation we try to nullify the non-diagonal element, for example $t_{pq}^{(k-1)}$ so, that $t_{pq}^{(k)} = 0$. This is possible to produce by a transformation matrix \mathbf{S}_k . We can imagine the

transformation geometrically by rotation of the axes in the plane by the angle \mathcal{G}_k . If we want to nullify the element on the row p and column q the transformation matrix will be

$$\mathbf{S}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos \mathcal{G}_k & 0 & \sin \mathcal{G}_k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ & & -\sin \mathcal{G}_k & 0 & \cos \mathcal{G}_k & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By using this transformation matrix we make the necessary transformation

$$\mathbf{T}_k = \mathbf{S}_k^T \mathbf{T}_{k-1} \mathbf{S}_k$$

After the necessary multiplication we get:

$$\begin{aligned} t_{jp}^{(k)} = t_{pj}^{(k)} = t_{pj}^{(k-1)} \cos \mathcal{G}_k - t_{qj}^{(k-1)} \sin \mathcal{G}_k & \quad \text{for } j \neq p \quad a \quad j \neq q \\ t_{jq}^{(k)} = t_{qj}^{(k)} = t_{pj}^{(k-1)} \sin \mathcal{G}_k + t_{qj}^{(k-1)} \cos \mathcal{G}_k & \end{aligned} \quad (3.2.18)$$

$$\begin{aligned} t_{ip}^{(k)} = t_{pi}^{(k)} = t_{pi}^{(k-1)} \cos \mathcal{G}_k - t_{qi}^{(k-1)} \sin \mathcal{G}_k & \quad \text{for } i \neq p \quad a \quad i \neq q \\ t_{iq}^{(k)} = t_{qi}^{(k)} = t_{pi}^{(k-1)} \sin \mathcal{G}_k + t_{qi}^{(k-1)} \cos \mathcal{G}_k & \end{aligned} \quad (3.2.19)$$

$$\begin{aligned} t_{pp}^{(k)} = t_{pp}^{(k-1)} \cos^2 \mathcal{G}_k + t_{qq}^{(k-1)} \sin^2 \mathcal{G}_k - 2t_{pq}^{(k-1)} \sin \mathcal{G}_k \cos \mathcal{G}_k \\ t_{qq}^{(k)} = t_{pp}^{(k-1)} \sin^2 \mathcal{G}_k + t_{qq}^{(k-1)} \cos^2 \mathcal{G}_k + 2t_{pq}^{(k-1)} \sin \mathcal{G}_k \cos \mathcal{G}_k \end{aligned} \quad (3.2.20)$$

$$t_{pq}^{(k)} = t_{qp}^{(k)} = \frac{1}{2} \left(t_{pp}^{(k-1)} - t_{qq}^{(k-1)} \right) \sin 2\mathcal{G}_k + t_{pq}^{(k-1)} \cos 2\mathcal{G}_k = 0$$

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \quad \text{for } i \neq p \quad a \quad j \neq q$$

While we nullify the element $t_{pq}^{(k)}$ the third of equation (3.2.20) must be zero. Therefore

$$tg 2\mathcal{G}_k = -\frac{2t_{pq}^{(k-1)}}{t_{pp}^{(k-1)} - t_{qq}^{(k-1)}} \Rightarrow \mathcal{G}_k = \frac{1}{2} \arctg \frac{2t_{pq}^{(k-1)}}{t_{pp}^{(k-1)} - t_{qq}^{(k-1)}} \quad (3.2.21)$$

By this manner we continue in all next steps. The number of orthogonal transformations needed to achieve the diagonal form is infinite. In practice, however, the process can be stopped when the non-diagonal terms tend to zero with the required accuracy. Usually we finished the process when the norm of non-diagonal terms reach the required accuracy or the average value of maximum non-diagonal element is smaller then some specified value:

$$\sqrt{\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^n a_{ij}^2 / (n(n-1)) \right)} = q_2$$

The eigenvectors of matrix \mathbf{A} are given by the orthogonal column matrix \mathbf{S}_k . If we define $\mathbf{R}_k = \mathbf{S}_1 \mathbf{S}_2 \dots \mathbf{S}_k = \mathbf{R}_{k-1} \mathbf{S}_k$ we can use the elements from the last step and determine the elements of eigenvector:

$$\begin{aligned} r_{ip}^{(k)} &= r_{ip}^{(k-1)} \cos \mathcal{G}_k - r_{iq}^{(k-1)} \sin \mathcal{G}_k \\ r_{iq}^{(k)} &= r_{ip}^{(k-1)} \sin \mathcal{G}_k + r_{iq}^{(k-1)} \cos \mathcal{G}_k \\ r_{ij}^{(k)} &= r_{ij}^{(k-1)} \quad \text{pro } j \neq p, q \end{aligned} \quad (3.2.22)$$

On the beginning of the iteration $\mathbf{R}_0 = \mathbf{E}$. The algorithm of the solution is in Tab. 3.3.

3.2.1.2 Symmetrisation of a matrix – Choleski algorithm

Jacobi's method is applicable only for symmetric matrices. In mechanical systems $\mathbf{A} = \mathbf{M}^{-1} \mathbf{K}$. This matrix is not symmetric even if matrices \mathbf{M} and \mathbf{K} are symmetric. If we want to use Jacobi's method it is necessary the matrix \mathbf{A} change to symmetric one. Because the mass matrix is positive definite, it can be factorized into a product of lower triangular matrix \mathbf{L} and its transposed counterpart:

$$\mathbf{M} = \mathbf{L}^T \mathbf{L} \quad (3.2.23)$$

We use the Choleski triangularization algorithm. If the mass matrix is diagonal it must be

$$\mathbf{L} = \mathbf{L}^T = \text{diag}[\sqrt{m_i}] \quad \text{and} \quad \mathbf{L}^{-1} = \mathbf{L}^{-T} = \text{diag}\left[\frac{1}{\sqrt{m_i}}\right] \quad (3.2.24)$$

For the eigenproblem it is valid

$$\mathbf{M}^{-1} \mathbf{K} \mathbf{v} = \lambda \mathbf{v} \quad (3.2.25)$$

Substituting in this equation for \mathbf{M} from (3.2.23) and define

$$\mathbf{v} = \mathbf{L}^{-1} \mathbf{y} \quad (3.2.26)$$

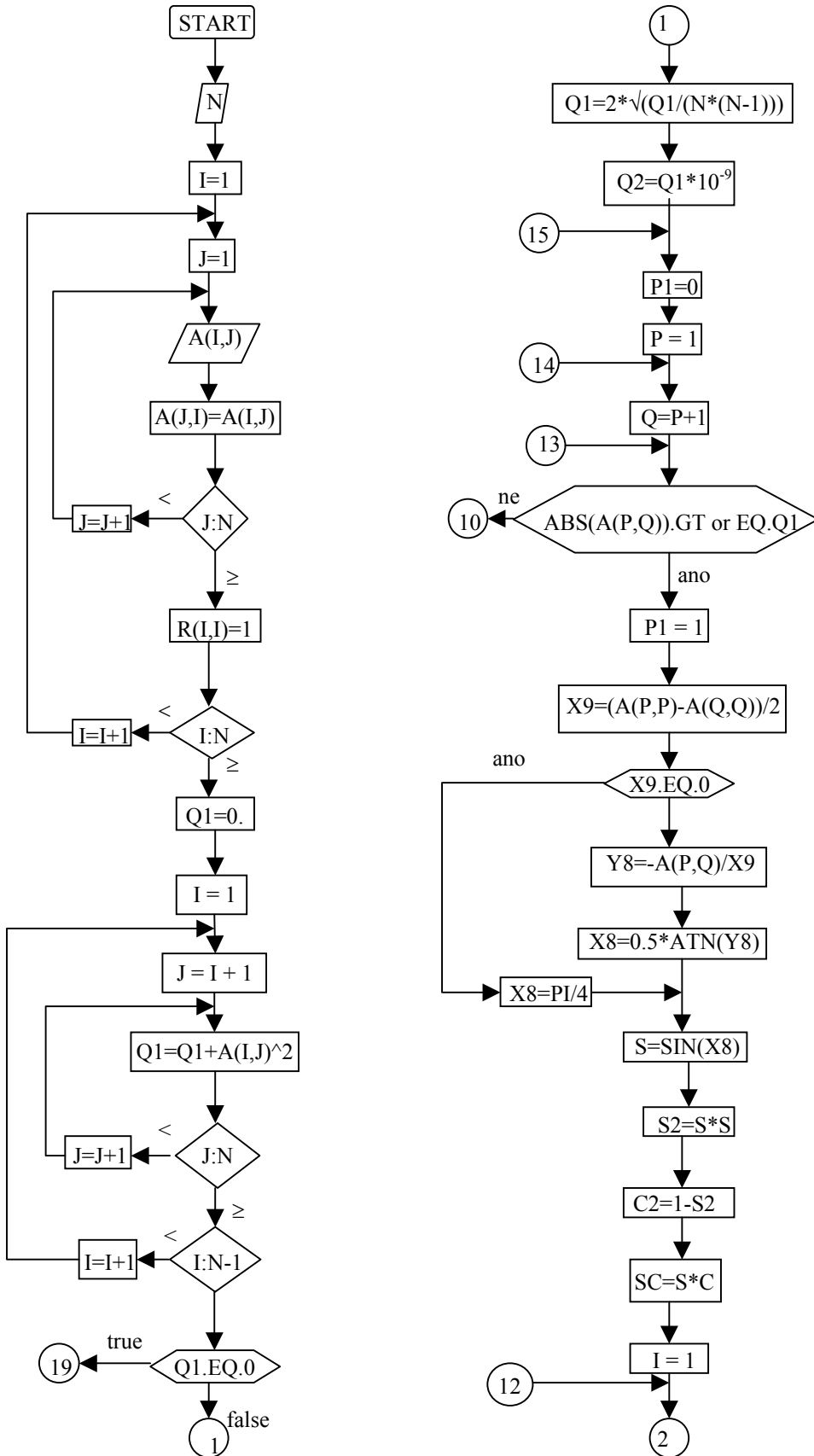
we get

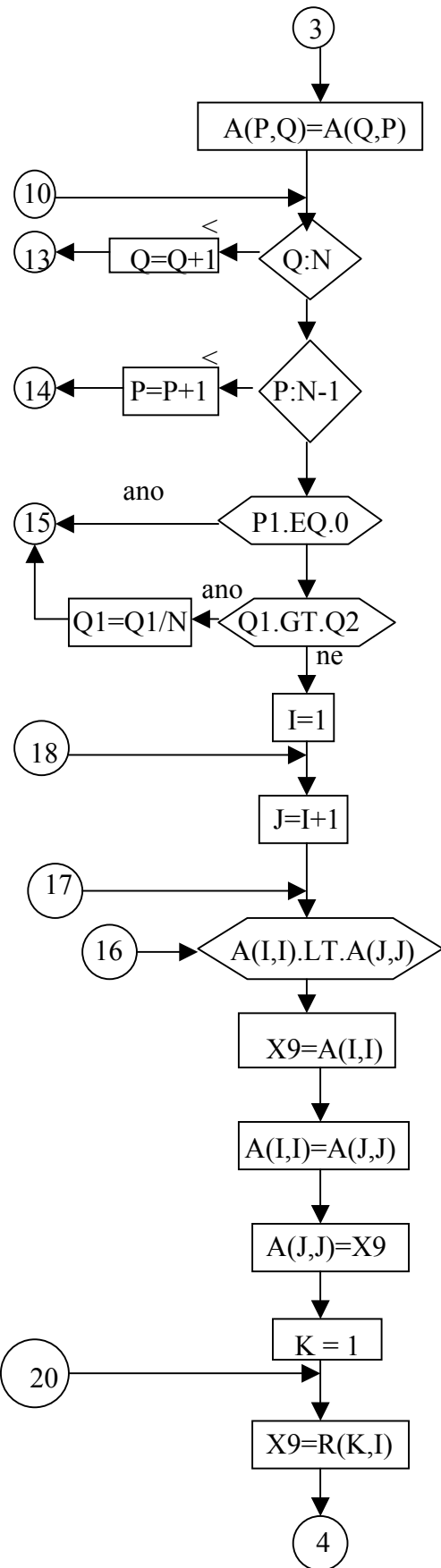
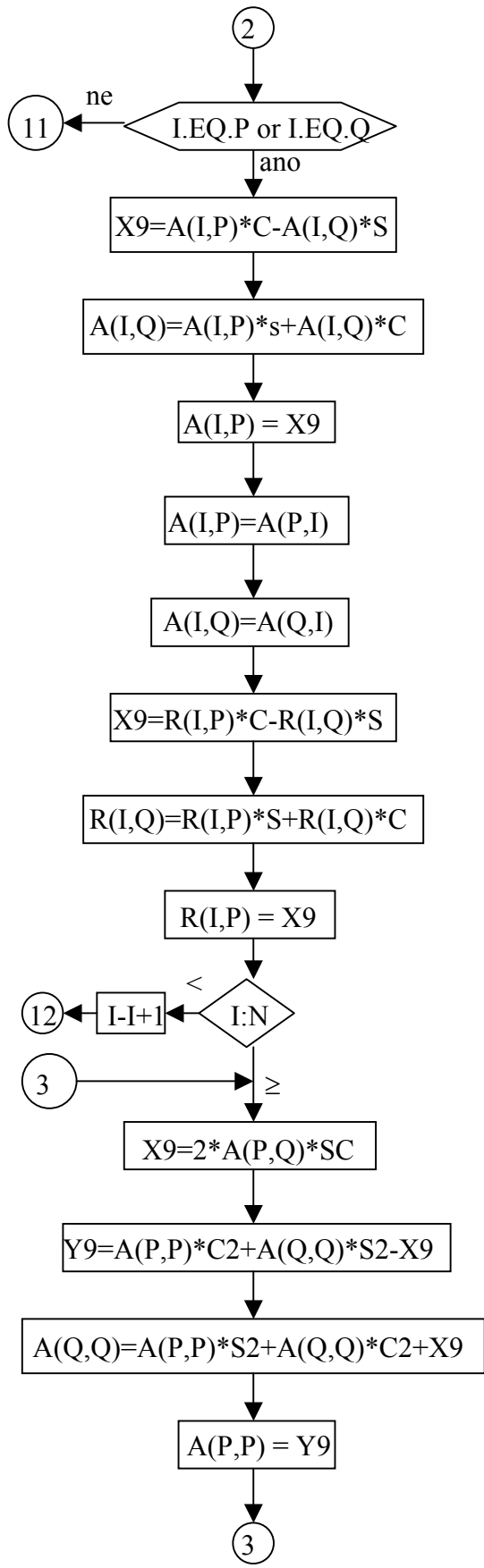
$$\mathbf{L}^{-T} \mathbf{L}^{-1} \mathbf{K} \mathbf{L}^{-1} \mathbf{y} = \lambda \mathbf{L}^{-1} \mathbf{y}$$

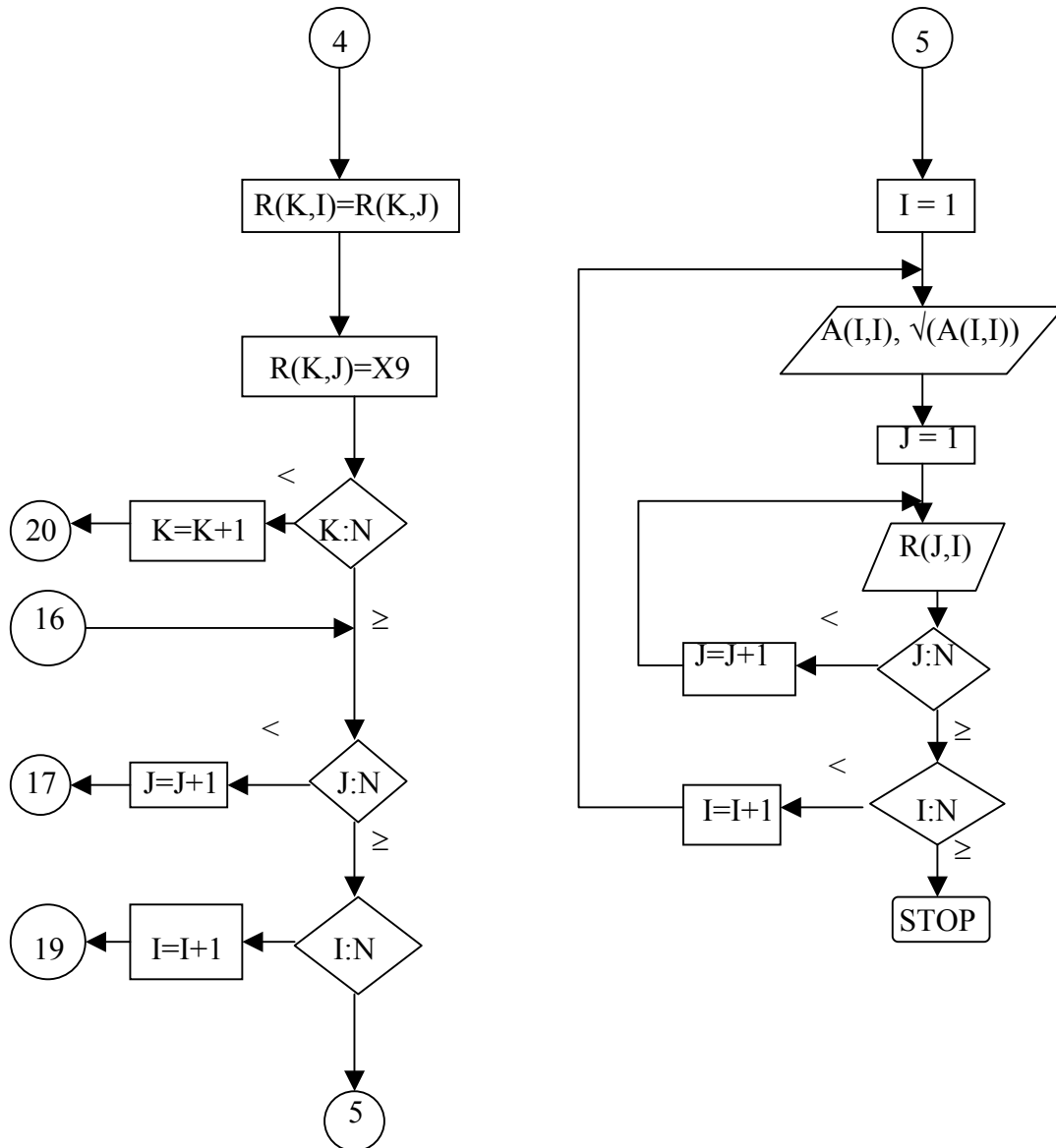
After multiplying this equation by matrix \mathbf{L} it will be obtained :

$$\mathbf{L}^{-1} \mathbf{K} \mathbf{L}^{-1} \mathbf{y} = \lambda \mathbf{y} \quad (3.2.27)$$

Tab. 3.3. Jacobi's algorithm







Because $\mathbf{L}^{-1}\mathbf{KL}^{-1}$ is symmetric it is possible to apply the Jacobi's method. Because the similarity transform was produced, the natural frequencies do not change. To obtain right vectors it is necessary apply the transformation according (3.2.26).

3.2.1.4 Reduction to tri-diagonal form – Householder's method

Householder's method is a successive transformation method for reducing the initial matrix to a tri-diagonal form in $(n - 2)$ steps. Unlike Jacobi's method, it implies a finite number of transformations. Because of their low cost, tri-diagonalization methods are widely used for solving moderate-size problems.

We aim to construct successive orthogonal transformation matrices $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_r$ so that the matrix resulting from the r^{th} transformation:

$$\mathbf{A}_k = \mathbf{P}_k^T \mathbf{P}_{k-1}^T \dots \mathbf{P}_1^T \mathbf{A} \mathbf{P}_1 \dots \mathbf{P}_{k-1} \mathbf{P}_k = \mathbf{P}_k^T \mathbf{A}_{k-1} \mathbf{P}_k \quad (3.2.28)$$

takes the form

$$\mathbf{A}_r = \left[\begin{array}{cccc} ** & & & \\ *** & & & \\ *** & & & 0 \\ \ddots & \ddots & & \\ 0 & ***** & & \\ & ***** & & \\ & ***** & & \end{array} \right] \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}} \right\} \begin{array}{c} \boxed{\text{r rows}} \end{array}$$

$\underbrace{\hspace{10em}}_{r \text{ columns}}$

To do so, we consider elementary transformation

$$\mathbf{v}_k^T \mathbf{v}_k = 1 \quad (3.2.29)$$

$$\mathbf{P}_k = \mathbf{E} - 2\mathbf{v}_k \mathbf{v}_k^T \quad (3.2.30)$$

By definition, these are orthogonal and symmetric. The transformations are constructed so as to leave the $(k - 1)$ first rows and columns of \mathbf{A}_{k-1} unchanged and to put to zero the non-tridiagonal terms of line and column number k ; This condition is fulfilled by equating to zero the first k terms off \mathbf{v}_k . We suppose that the vector \mathbf{v}_k has its first $(k-1)$ elements equal to zero

$$\mathbf{v}_k^T = \underbrace{[0 \ 0 \ 0 \ \dots \ * \ * \ * \ \dots \ *]}_{(k-1)} \quad (3.2.31)$$

The task is now to determine the elements v_{kk} to v_{kn} so that must be valid (3.2.29) and $(n - k)$ non-tridiagonal elements in the row and column of matrix \mathbf{A}_k are equal to zero. The proceeding will be so, that we put

$$S = \sum_{j=k}^n a_{k-1,j}^2 \quad (3.2.32)$$

$$v_{k,k}^2 = \frac{1}{2} \left[1 \pm \left(a_{k-1,k} / \sqrt{S} \right) \right] \quad (3.2.33)$$

$$v_{k,j} = \pm a_{k-1,j} / (2v_{k,k} \sqrt{S}) \quad j = k+1, \dots, n \quad (3.2.34)$$

We choose the sign in (3.2.33) and (3.2.34) so, that the solution will be most precise. Because v_{kk} in (3.2.34) is in the denominator it has to be as great as possible. Therefore we choose the sign in (3.2.33) equal to the sign of $a_{k-1,k}$ and the same sign will be used in (3.2.34). So is given the principle of the solution. For maximum effectiveness we use the following proceeding:

From (3.2.28) we determine

$$\mathbf{A}_{k-1} \mathbf{P}_k = \mathbf{A}_{k-1} (\mathbf{E} - 2\mathbf{v}_k \mathbf{v}_k^T) \quad (3.2.35)$$

We choose the notation $\mathbf{w}_k = \mathbf{A}_{k-1} \mathbf{v}_k$ and we write

$$\mathbf{A}_{k-1} \mathbf{P}_k = \mathbf{A}_{k-1} - 2\mathbf{w}_k \mathbf{v}_k^T \quad (3.2.36)$$

We determine from (3.2.28)

$$\mathbf{A}_k = \mathbf{P}_k^T \mathbf{A}_{k-1} \mathbf{P}_k = \mathbf{A}_{k-1} - 2\mathbf{v}_k \mathbf{q}_k^T - 2\mathbf{q}_k \mathbf{v}_k^T \quad (3.2.37)$$

where it is

$$\mathbf{q}_k = \mathbf{w}_k - \mathbf{v}_k^T \mathbf{w}_k \mathbf{v}_k$$

3.2.1.5 Rayleigh's quotient

Very often is sufficient to know only one natural frequency, usually the lowest one. We show now how to do it. We start from equation (3.2.3)

$$\mathbf{K} \mathbf{v}_r = \Omega_r^2 \mathbf{M} \mathbf{v}_r$$

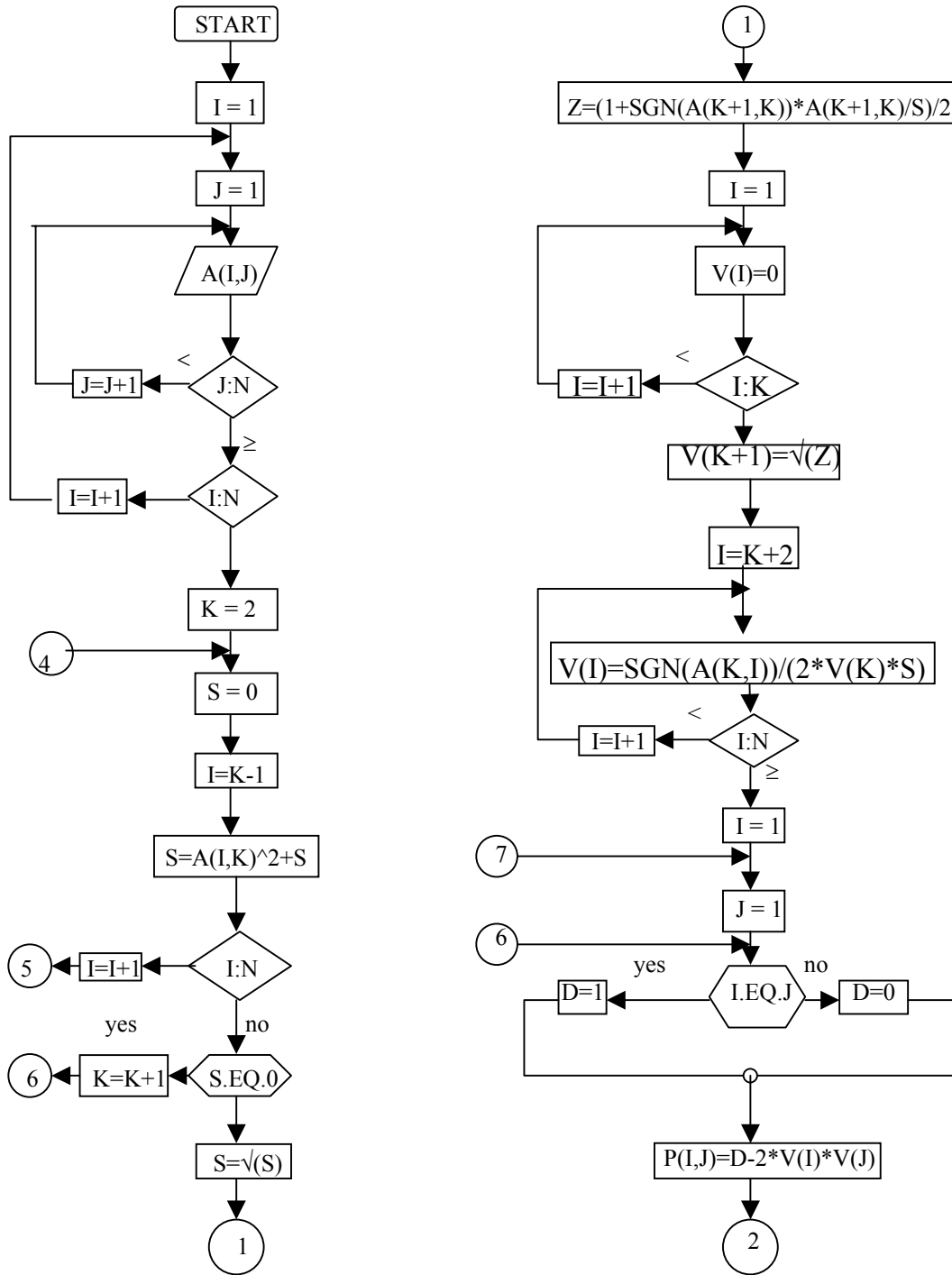
We multiply this equation from right hand side by vector \mathbf{v}_r^T . Both sides of the equation are then scalars and we may determine $\lambda_r = \Omega_r^2$:

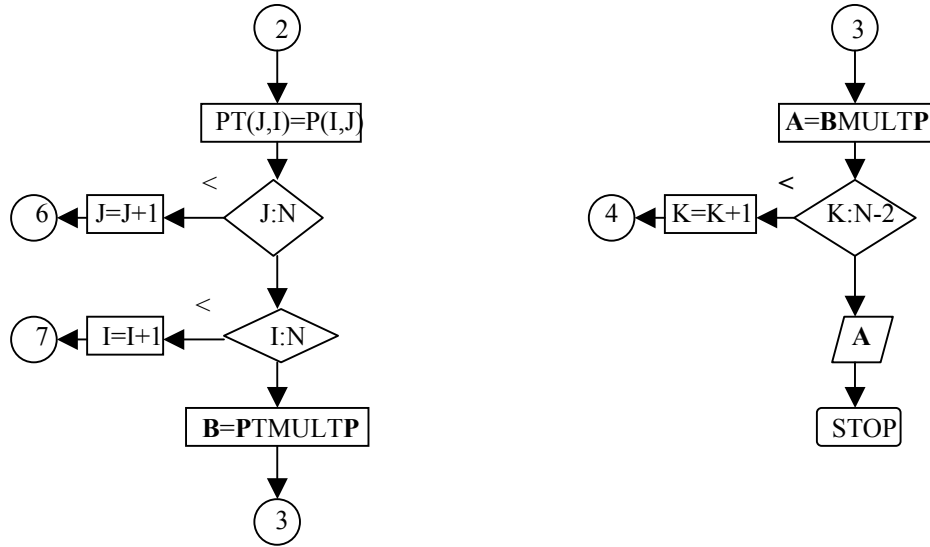
$$\lambda_r = \frac{\mathbf{v}_r^T \mathbf{K} \mathbf{v}_r}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \quad (3.2.38)$$

\mathbf{v}_r are the modal vectors of free non-damped vibration. In praxis the modal vectors $\bar{\mathbf{v}}_r$ are judged so (3.2.38) obtains the form

$$\bar{\lambda}_r = \frac{\bar{\mathbf{v}}_r^T \mathbf{K} \bar{\mathbf{v}}_r}{\bar{\mathbf{v}}_r^T \mathbf{M} \bar{\mathbf{v}}_r} \quad (3.2.39)$$

Tab. 3.4 The algorithm of Householder's method





This equations determines so called *Rayleigh's quotient* which has following properties:

1. If the judged vector \bar{v}_r is judged with some accuracy the quotient is of one order more accurate
2. If the vector \bar{v}_r is equal to the exact value v_r the Rayleigh's quotient is equal to the real value of the square of natural frequency Ω_o .
3. If the quotient reach the values of all modes, then it will be in the interval of accurate natural frequencies.

If we multiply the nominator and denominator of the equation (3.2.38) ve may also write

$$\lambda_r = \frac{E_{Pr}}{E_{Kr}^*} \quad (3.2.40)$$

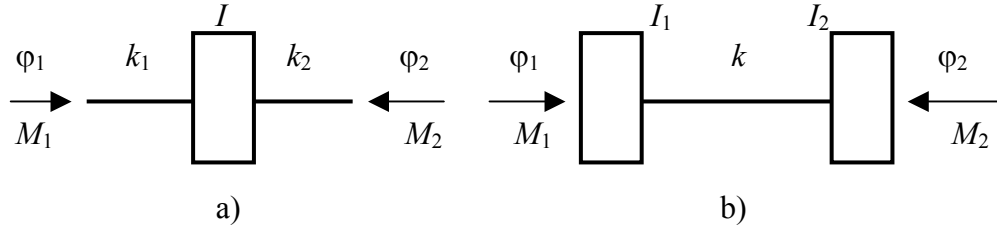
E_{Kr}^* is the unit kinetic energy of mode r ($\Omega_{or} = 1$)

3.2.2 Reduction of number of degree of freedom

If the mechanical system has many degree of freedom and we are not interesting about the higher natural frequencies we reduce the number of freedom. Always must be satisfied the condition that the natural frequencies have to be equal to the frequencies of the original mechanical system. In next text we show two very often used methods of reduction.

3.2.2.1 Reduction by transformation of the mechanical model

This method uses step by step transformation of the mechanical model. Each system is possible is possible to divide on a raw of separated parts of two kinds (Fig.3.9)



Obr. 3.9

By free vibration transfer the elastic part of the mechanical system the torque M_i . Therefore we suppose that the system is loaded by external harmonic load in points 1 and 2..

The system a)

$$\begin{aligned} I\ddot{\varphi} + k_1(\varphi_1 - \varphi) + k_2(\varphi - \varphi_2) &= 0 \\ k_1(\varphi_1 - \varphi) &= M_1 \\ k_2(\varphi_2 - \varphi) &= -M_2 \end{aligned}$$

We substitute the harmonic components $\varphi = \varphi_m e^{i\omega t}$, $M_i = M_{0i} e^{i\omega t}$, and from the second equation we determine

$$\varphi_m = \varphi_{m1} - \frac{M_{01}}{k_1}$$

The previous set of equations obtains the form

$$\begin{aligned} -I\omega^2 \varphi_m + k_1(\varphi_m - \varphi_{m1}) + k_2(\varphi_m - \varphi_{m2}) &= 0 \\ (k_1 + k_2 - I\omega^2)\varphi_m - k_1\varphi_{m1} - k_2\varphi_{m2} &= 0 \\ (k_1 + k_2 - I\omega^2)\left(\varphi_{m1} - \frac{M_{01}}{k_1}\right) - k_1\varphi_{m1} - k_2\varphi_{m2} &= 0 \end{aligned}$$

We eliminate φ_m

$$\begin{aligned} k_2\varphi_{m1} + \frac{M_{01}I}{k_1}\omega^2 - k_2\varphi_{m2} - \frac{k_1 + k_2}{k_1}M_{01} - I\omega^2\varphi_{m1} &= 0 \\ \varphi_{m2} = \left(1 - \frac{I}{k_2}\omega^2\right)\varphi_{m1} - M_{01}\left(\frac{k_1 + k_2}{k_1k_2} - \frac{I}{k_1k_2}\omega^2\right) \end{aligned}$$

Using the natural circular frequency of the system a)

$$\Omega_{0a} = \sqrt{\frac{k_1 + k_2}{I}}$$

we obtain

$$\begin{aligned}\varphi_{m_2} &= \left(1 - \frac{I}{k_2} \omega^2\right) \varphi_{m_1} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right) \left(1 - \frac{\omega^2}{\Omega_{0a}^2}\right) M_{01} \\ M_{02} &= I \omega^2 \varphi_{m_1} + \left(1 - \frac{I \omega^2}{k_1}\right) M_{01}\end{aligned}\tag{3.2.41}$$

The system b)

The equations of motion are

$$\begin{aligned}I_1 \ddot{\varphi}_1 + k(\varphi_1 - \varphi_2) &= M_1 \\ I_2 \ddot{\varphi}_2 + k(\varphi_2 - \varphi_1) &= M_2\end{aligned}$$

We suppose $\varphi_i = \varphi_{mi} e^{i\omega t}$; $M_i = M_{0i} e^{i\omega t}$. Then

$$\begin{aligned}-I_1 \omega^2 \varphi_{m_1} + k(\varphi_{m_1} - \varphi_{m_2}) &= M_{01} \\ -I_2 \omega^2 \varphi_{m_2} + k(\varphi_{m_2} - \varphi_{m_1}) &= M_{02}\end{aligned}$$

From here

$$\begin{aligned}\varphi_{m_2} &= \left(1 - \frac{I_1}{k} \omega^2\right) \varphi_{m_1} - \frac{1}{k} M_{01} \\ M_{02} &= (I_1 + I_2) \omega^2 \left(1 - \frac{\omega^2}{\Omega_{0b}^2}\right) \varphi_{m_1} + \left(1 - \frac{I_2}{k} \omega^2\right) M_{01}\end{aligned}$$

The natural frequency is given by

$$\Omega_{0b} = \sqrt{\frac{I_1 + I_2}{I_1 I_2} k}$$

Finally we get

$$\begin{aligned}\varphi_{m_2} &= \left(1 - \frac{I_1}{k} \omega^2\right) \varphi_{m_1} - \frac{1}{k} M_{01} \\ M_{02} &= (I_1 + I_2) \omega^2 \left(1 - \frac{\omega^2}{\Omega_{0b}^2}\right) \varphi_{m_1} + \left(1 - \frac{I_2}{k} \omega^2\right) M_{01}\end{aligned}\tag{3.2.42}$$

The principle of this method is in mutual transformation of one system by the second one.

The transformation of the system a) by the system b)

In this case is valid

$$\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k' = \frac{k_1 k_2}{k_1 + k_2}\tag{3.2.43}$$

Comparing terms by φ_{m_1} and M_{01} we get

$$I'_1 = \frac{k_1}{k_1 + k_2} I \quad \text{and} \quad I'_2 = \frac{k_2}{k_1 + k_2} I \quad (3.2.44)$$

The transformation of the system b) by the system a)

In this case it is

$$I'' = I_1 + I_2 \quad (3.2.45)$$

Comparing (3.2.41) and (3.2.42) we get

$$k''_1 = \frac{I_1 + I_2}{I_2} k \quad \text{and} \quad k''_2 = \frac{I_1 + I_2}{I_1} k \quad (3.2.46)$$

Using (3.2.43) and (3.2.44) the equations (3.2.42) obtain the form

$$\begin{aligned} \varphi_{m2} &= \left(1 - \frac{I}{k_2} \omega^2\right) \varphi_{m1} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right) M_{01} \\ M_{02} &= I \omega^2 \left(1 - \frac{\omega^2}{\Omega_{0a}^2}\right) \varphi_{m1} + \left(1 - \frac{I}{k_1} \omega^2\right) M_{01} \end{aligned} \quad (3.2.47)$$

and using (3.2.45) and (3.2.46) will be

$$\begin{aligned} \varphi_{m2} &= \left(1 - \frac{I_1}{k} \omega^2\right) \varphi_{m1} - \frac{1}{k} \left(1 - \frac{\omega^2}{\Omega_{0b}^2}\right) M_{01} \\ M_{02} &= (I_1 + I_2) \omega^2 \varphi_{m1} + \left(1 - \frac{I_2}{k} \omega^2\right) M_{01} \end{aligned} \quad (3.2.48)$$

Comparing (3.2.47) with (3.2.41) we see that the difference is only in members $\left(1 - \frac{\omega^2}{\Omega_{0a}^2}\right)$. If is $\frac{\omega}{\Omega} \leq 2,5$ the transformation is with sufficient accuracy. The process is shown on Fig. 3.10.

The algorithm is following:

1. We determine the subsystem with maximal natural frequency

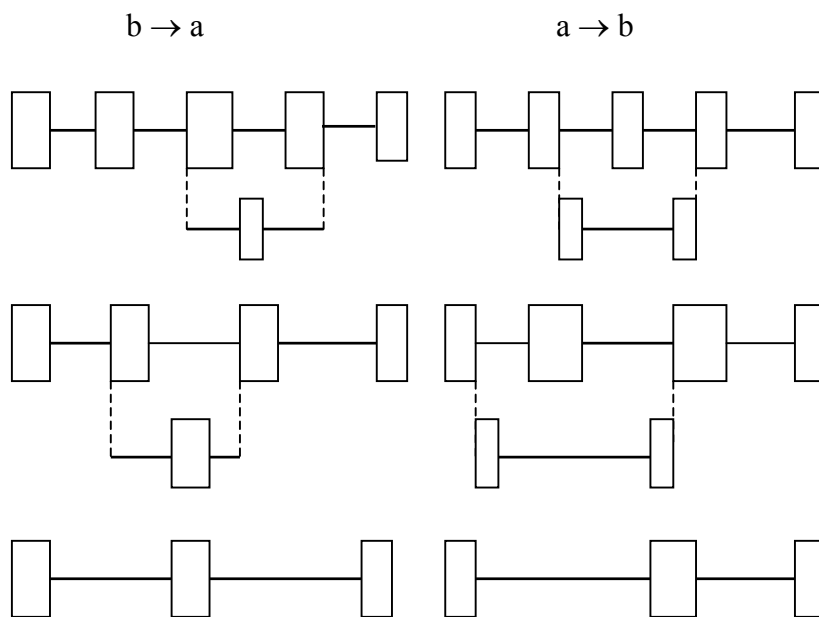
$$\text{a) } \Omega_{0a} = \sqrt{\frac{k_1 + k_2}{I}} \quad \text{b) } \Omega_{0b} = \sqrt{\frac{I_1 + I_2}{I_1 I_2} k}$$

1. We make the reduction of the system – change of subsystems:

$$\text{a} \rightarrow \text{b} \quad k' = \frac{k_1 k_2}{k_1 + k_2}, \quad I'_1 = \frac{k_1}{k_1 + k_2} I, \quad I'_2 = \frac{k_2}{k_1 + k_2}$$

$$\text{b} \rightarrow \text{a} \quad I'' = I_1 + I_2, \quad k''_1 = \frac{I_1 + I_2}{I_2} k, \quad k''_2 = \frac{I_1 + I_2}{I_1} k$$

The procedure repeats as long as we get the required number of degrees of freedom. The advantage of this method is in objective. We know how the original system was changed. However, it does not suit for systems with many degrees of freedom.



Obr. 3.10

3.2.2.2 Lanczos – Ojalvo method of reduction

The mechanical systems of high degree of freedom is better to use some method that works automatically according a fixed algorithm. One of such method is *Lanczos method*. The principle of this method consists of generating a subspace including the system fundamental eigensolutions by inverse iteration on one starting vector.

Let we consider the equation

$$\mathbf{K}\mathbf{v} = \lambda\mathbf{M}\mathbf{v} \quad (3.2.49)$$

The matrices are positive definite symmetric matrices of order n . Lanczos method reduce the system so, that the chosen number m of calculated natural frequencies $\lambda = \omega^2$ agrees with first m numbers of the original system.

Let we choose $m < n$ and form the reduced matrix $\mathbf{R} = [\mathbf{r}_i]$ of order (n, m) . \mathbf{r}_i are so called Lanczos vectors of order $(n, 1)$. Then we define vector \mathbf{y} $(m, 1)$ so that it is valid

$$\mathbf{v} = \mathbf{R}\mathbf{y} \quad (3.2.50)$$

Substituting this equation into (3.2.49) we get

$$\mathbf{K}\mathbf{R}\mathbf{y} = \lambda\mathbf{M}\mathbf{R}\mathbf{y}$$

We multiply the equation from left hand side by transposed matrix \mathbf{R}^T :

$$(\mathbf{R}^T\mathbf{K}\mathbf{R})\mathbf{y} = \lambda(\mathbf{R}^T\mathbf{M}\mathbf{R})\mathbf{y} \quad (3.2.51)$$

By this way is the problem reduced on the order m . The reduced matrix $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m]$ will be determined by using the Lanczos mechanism so that the matrix $\mathbf{R}^T\mathbf{M}\mathbf{R}$ is a unit diagonal matrix and matrix $\mathbf{R}^T\mathbf{K}\mathbf{R}$ will be tri-diagonal. Then (3.2.51) is possible to write

$$\mathbf{R}^T\mathbf{K}\mathbf{R}\mathbf{y} = \lambda\mathbf{y} \quad (3.2.52)$$

So the original problem is reduced on the problem of symmetric matrix, which is not only reduced to the order m but more over is tri-diagonal.

The algorithm of Lanczos improved by Ojalvo is described:

1. The product $\mathbf{M}^{-1}\mathbf{K}$ is calculated
2. The judgement $\bar{\mathbf{r}}_1$ of the vector \mathbf{r}_1 is done. It was proved that the best results are obtained by using the generator of random numbers in the interval $(0, 1)$.
3. Now follows the calculation of vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ in steps form 1 to m by following procedure
4. $\beta_i^2 = \bar{\mathbf{r}}_i^T \mathbf{M} \bar{\mathbf{r}}_i$
5. $\mathbf{r}_i = \frac{\bar{\mathbf{r}}_i}{\beta_i}$ (By this step is made the normalization)
6. $\alpha_i = \mathbf{r}_i^T \mathbf{K} \mathbf{r}_i$
7. For $i = m$ one continue from step 13.
8. For $i = 1$: $\bar{\mathbf{r}}_2 = (\mathbf{M}^{-1}\mathbf{K})\mathbf{r}_1 - \alpha_1\mathbf{r}_1$
for $i > 1$ ($i = 1, 2, \dots, m-1$) $\bar{\mathbf{r}}_{i+1} = (\mathbf{M}^{-1}\mathbf{K})\mathbf{r}_i - \alpha_i\mathbf{r}_i - \beta_i\mathbf{r}_{i-1}$
9. It will be performed the loop of orthogonalisation in steps $s = 0$ up to s . In these steps will be made the correction of the vector \mathbf{r}_{i+1} so that it will be orthogonal with vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i$. $s = 0$
10. $\bar{\mathbf{r}}_{i+1}^{s+1} = \bar{\mathbf{r}}_{i+1}^s - \sum_{j=1}^i (\mathbf{r}_{i+1-j}^T \mathbf{M} \bar{\mathbf{r}}_{i+1}^s) \mathbf{r}_{i+1-j}$

11. If in the sum is $\mathbf{r}_{i+1-j}^T \mathbf{M} \bar{\mathbf{r}}_{i+1}^s \rangle \varepsilon$ then $s = s + 1$ and we go to the position 10.

ε is the accuracy, for example 10^{-9}

12. After successful loop ($\mathbf{r}_{i+1-j}^T \mathbf{M} \bar{\mathbf{r}}_{i+1}^s \leq \varepsilon$) we use for next calculation $\mathbf{r}_{i+1} = \bar{\mathbf{r}}_{i+1}^{s+1}$ and we continue from position 4

13. All vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ are determined. Simultaneously are determined the necessary values α_i, β_i for $i = 1, 2, \dots, m$, which are used to perform the matrix

$$\mathbf{R}^T \mathbf{K} \mathbf{R} = \begin{bmatrix} \alpha_1 & \beta & 0 & \cdot & \cdot & 0 \\ \beta_2 & \alpha_2 & \beta_3 & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & \beta_{m-1} & \alpha_{m-1} & \beta_m \\ \cdot & \cdot & \cdot & \cdot & \beta_m & \alpha_m \end{bmatrix}$$

14. The eigenvalues and eigenvectors using some standard method (Jacobi).

15. We determine the modal vector of the original matrix

$$\mathbf{v} = \mathbf{R} \mathbf{y}$$

A detailed analysis of the method shows that the convergence is extremely fast. Since it is based on the inverse iteration process applied to only one starting vector, it can be applied to very large systems with low cost and storage requirements.

3.2.3 Free damped vibrations

The vibration of a system with damping is described by the equation

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{B} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{0} \quad (3.2.53)$$

The presence of the damping complicates considerably the solution of the problem and makes it more difficult to understand dynamic system behaviour.

\mathbf{B} is a square symmetric matrix of order n . The construction of this matrix is very difficult because we do not know nor the arrangement of linear dampers nor its damping constancies.

Therefore we introduce the *proportional damping* described as a part of mass matrix and stiffness matrix

$$\mathbf{B} = \alpha \mathbf{M} + \beta \mathbf{K} \quad (3.2.54)$$

In this equation the term $\alpha \mathbf{M}$ represents the external damping and $\beta \mathbf{K}$ represents the internal material damping. By proportional damping the rule of orthogonality is simple

$$\mathbf{v}_r^T \mathbf{B} \mathbf{v}_s = 0 \quad \text{pro } r \neq s \quad (3.2.55)$$

Let us consider the conservative system associated with the real system. It is governed by the equation of motion $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$ and its modal vectors are \mathbf{v}_r and Ω_{0r} .

Let us try as in the undamped case, to solve the system of equations (3.2.53) through modal vectors \mathbf{v}_r of the associated conservative system

$$\mathbf{q} = \sum_r C_r e^{\lambda_r t} \mathbf{v}_r \quad (3.2.56)$$

Substituting (3.2.56) into (3.2.53) we get

$$\mathbf{M} \sum_r \lambda_r^2 C_r e^{\lambda_r t} \mathbf{v}_r + \mathbf{B} \sum_r \lambda_r C_r e^{\lambda_r t} \mathbf{v}_r + \mathbf{K} \sum_r C_r e^{\lambda_r t} \mathbf{v}_r = \mathbf{0}$$

Let us multiply this equation from left hand side by \mathbf{v}_r^T and use the conditions of orthogonality:

$$C_r \left(\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r \lambda_r^2 + \mathbf{v}_r^T \mathbf{B} \mathbf{v}_r \lambda_r + \mathbf{v}_r^T \mathbf{K} \mathbf{v}_r \right) e^{\lambda_r t} = 0 \quad \text{for } r=1,2,\dots,n \quad (3.2.57)$$

Because the solution must be accepted for any time, the expression in brackets has to be zero.

We use the previous notation

$$m_{yr} = \mathbf{v}_r^T \mathbf{M} \mathbf{v}_r; \quad k_{yr} = \mathbf{v}_r^T \mathbf{K} \mathbf{v}_r$$

and the damping matrix \mathbf{B} describe by (3.2.54) we obtain n independent equations

$$m_{yr} \lambda_r^2 + (\alpha m_{yr} + \beta k_{yr}) \lambda_r + k_{yr} = 0 \quad (r=1,2,\dots,n) \quad (3.2.58)$$

The roots of (3.2.58) are

$$(\lambda_r)_{1,2} = -\delta_r \pm i\Omega_r \quad (3.2.59)$$

In (3.2.59) means

$$\delta_r = \frac{\alpha m_{yr} + \beta k_{yr}}{2m_{yr}} \quad (3.2.60)$$

$$\Omega_r = \sqrt{\Omega_{0r}^2 - \delta_r^2}$$

$$\Omega_{0r} = \sqrt{\frac{k_{yr}}{m_{yr}}} \quad (3.2.61)$$

The general solution of (3.2.56) is

$$\mathbf{q} = \sum (C_{1r} e^{\lambda_{r1} t} + C_{2r} e^{\lambda_{r2} t}) \mathbf{v}_r \quad (3.2.62)$$

If $\Omega_{0r} > \delta_r$ the roots λ_r will be a complex values and the resulting motion will be periodic

$$\mathbf{q} = \sum_r e^{-\delta_r t} (A_r \cos \Omega_r t + B_r \sin \Omega_r t) \mathbf{v}_r \quad (3.2.63)$$

$$\mathbf{q} = \sum_r C_r e^{-\delta_r t} \sin(\Omega_r t + \varphi_r) \mathbf{v}_r \quad (3.2.64)$$

$C_{1r}, C_{2r}; A_r, B_r; \text{ or } C_r, \varphi_r$

are integration constants determined from initial conditions ($\mathbf{q} = \mathbf{q}_0, \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}_0$)

The proportional coefficients α and β are determined experimentally. From (3.2.60) we get

$$b_r = \frac{1}{2} \left(\frac{\alpha}{\Omega_{0r}} + \beta \Omega_r \right)$$

Because there are two unknowns two measurements must be made by two different natural frequencies. Because the higher modes are difficult to excite we use other procedure. From practice we know that extreme damping is by lowest frequency we put the derivative of the last equation to zero.

$$\frac{db_r}{d\Omega_r} = \frac{1}{2} \left(-\frac{\alpha}{\Omega_{0r}^2} + \beta \right) = 0$$

Now we can use this equation and one measurement by first natural frequency.

The coefficients are given by

$$\begin{aligned} \alpha &= \Omega_{0r} b_r \\ \beta &= \frac{b_r}{\Omega_{0r}} \end{aligned} \quad (3.2.65)$$

3.2.4 Forced response of mechanical systems

In this case we suppose the equation of motion of the mechanical system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}(t) \quad (3.2.66)$$

We have a differential equation of second order with right said. The solution of it is composed of the homogenous solution and a particular one

$$\mathbf{q} = \mathbf{q}_h + \mathbf{q}_p$$

The homogenous part is given by (3.2.62), (3.2.63) or (3.2.64).

The particular solution depends of the character of the excited force.

3.2.4.1 The force is harmonic

By this excitation it is possible to solve many practical cases. We suppose the excited vector of the force

$$\mathbf{Q}(t) = \mathbf{Q}_0 e^{i\omega t}$$

With respect to the harmonic right hand side of the equation we choose the particular solution

$$\tilde{\mathbf{q}}_p = \tilde{\mathbf{s}}e^{i\omega t}$$

After substituting in (3.2.66) we get

$$(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{B})\tilde{\mathbf{s}} = \mathbf{Q}_0$$

From this equation it is possible to get the complex vector $\tilde{\mathbf{s}}$

$$\tilde{\mathbf{s}} = (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{B})^{-1}\mathbf{Q}_0 \quad (3.2.67)$$

It is necessary to keep in mind that we have to obtain the inverse of a complex matrix.

The inversion of it is

$$(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{B})^{-1} = \frac{\mathbf{G}(\omega)}{\Delta(\omega)} \quad (3.2.68)$$

Here is

$$\begin{aligned} \mathbf{G}(\omega) &= \text{adj}(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{B}) \\ \Delta(\omega) &= \det(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{B}) \end{aligned}$$

The real part of the response are given

$$s_{0r} = \sqrt{(\text{Re}\{\tilde{s}_r\})^2 + (\text{Im}\{\tilde{s}_r\})^2} \quad \text{for } r = 1, 2, \dots, n \quad (3.2.69)$$

The phase is given by

$$\varphi_{pr} = \text{arctg} \frac{\text{Im}\{\tilde{s}_r\}}{\text{Re}\{\tilde{s}_r\}} \quad \text{for } r = 1, 2, \dots, n \quad (3.2.70)$$

The general solution can be written

$$\mathbf{q} = \underbrace{\sum_r [C_r e^{-\delta_r t} \sin(\Omega_r t + \varphi_r) \mathbf{v}_r + \mathbf{s}_r \sin(\omega t + \varphi_{pr})]}_{\text{homogenous solution}} \quad (3.2.71)$$

homogenous solution

The integration constants C_r and φ_r are determined from the initial conditions.

3.2.4.2 Exciting force is a general function of time

Often the exciting force is a general function of time. The particular solution of the equation (3.2.66) is supposed in the form

$$\mathbf{q}_p = \sum_r \mathbf{v}_r d_r(t) \quad (3.2.72)$$

\mathbf{v}_r is the modal vector of the free not-damped vibration. $d_r(t)$ is till now unknown function of time. Substituting (3.2.72) into (3.2.66) we get

$$\mathbf{M} \sum_r \mathbf{v}_r \ddot{d}_r(t) + \mathbf{B} \sum_r \mathbf{v}_r \dot{d}_r(t) + \mathbf{K} \sum_r \mathbf{v}_r d_r(t) = \mathbf{Q}(t)$$

Multiplying this equation by \mathbf{v}_r^T and using the conditions of orthogonality we obtain

$$\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r \ddot{d}_r(t) + \mathbf{v}_r^T \mathbf{B} \mathbf{v}_r \dot{d}_r(t) + \mathbf{v}_r^T \mathbf{K} \mathbf{v}_r d_r(t) = \mathbf{v}_r^T \mathbf{Q}(t) \quad \text{pro } r = 1, 2, \dots, n$$

We divide the equation by the main mass $m_{yr} = \mathbf{v}_r^T \mathbf{M} \mathbf{v}_r$ and denote

$$2\delta_r = \frac{\mathbf{v}_r^T \mathbf{B} \mathbf{v}_r}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} = \frac{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r \alpha + \mathbf{v}_r^T \mathbf{K} \mathbf{v}_r \beta}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \quad (3.2.73)$$

$$\Omega_{0r}^2 = \frac{\mathbf{v}_r^T \mathbf{K} \mathbf{v}_r}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \quad (3.2.74)$$

By this procedure we got n independent equations

$$\ddot{d}_r(t) + 2\delta_r \dot{d}_r(t) + \Omega_{0r}^2 d_r(t) = \frac{\mathbf{v}_r^T \mathbf{Q}(t)}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \quad \text{pro } r = 1, 2, \dots, n \quad (3.2.75)$$

This solution of this equation is given by the Duhamel integral

$$d_r(t) = \frac{\mathbf{v}_r^T}{\Omega_r m_{zr}} \int_0^t \mathbf{Q}(\tau) e^{-\delta_r(t-\tau)} \sin \Omega_r(t-\tau) d\tau$$

We substitute this solution into (3.2.72) and we obtain the particular solution of (3.2.66)

$$\mathbf{q}_p = \sum_r \frac{\mathbf{v}_r \mathbf{v}_r^T}{\Omega_r \mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \int \mathbf{Q}(\tau) e^{-\delta_r(t-\tau)} \sin \Omega_r(t-\tau) d\tau \quad (3.2.76)$$

In this equation it is

$$\Omega_r = \sqrt{\Omega_{0r}^2 - \delta_r^2}$$

The shown solution is valid also for the cases when $\alpha = 0$ or $\beta = 0$ or $\alpha = \beta = 0$. This solution is useful when if the matrix of dynamic stiffness is singular and the inverse of it is not possible. Equation (3.2.76) is possible to use also for a harmonic exciting force. In such a case we substitute in (3.2.75) $\mathbf{Q}(t) = \mathbf{Q}_0 e^{i\omega t}$ and the solution is supposed by

$$d_r = s_{0r} \sin(\omega t - \varphi_{pr})$$

We substitute into (3.2.72):

$$\mathbf{q}_p = \sum_r \mathbf{v}_r s_{0r} \sin(\omega t - \varphi_r) \quad (3.2.77)$$

The amplitude of harmonic response and phase is given by expressions

$$s_{0r} = \frac{1}{\sqrt{(\Omega_{0r}^2 - \omega^2)^2 + (2\delta_r\omega)^2}} \frac{\mathbf{v}_r^T \mathbf{Q}_0}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \quad (3.2.78)$$

$$\varphi_r = \arctg \frac{2\delta_r\omega}{\Omega_{0r}^2 - \omega^2} \quad (3.2.79)$$

When we use the computer the solution of response is very simple and comfortable even for the systems with extremely high number of degrees,

3.3 Vibrations of Continuous systems

So far, vibrating systems have been regarded as an assembly of discretized elements, namely rigid elements with inertia as their only physical property, linked together by springs and or dissipative elements, characterized by their stiffness and their damping coefficients but with no inertia

The number of degrees of freedom of a discrete systems is fixed by its number of masses and its mathematical model consists of a set of n coupled ordinary differential equations.

In practice, the representation of a physical system by a discrete model is usually an idealized view. In most cases, the main bodies, which compose a mechanical system are deformable, and the elastic elements which connect the main bodies has also their own inertia. Therefore each constituent of a system possesses simultaneously inertia, stiffness and damping properties. The mathematical model of a continuous system undergoing time dependent deformation used in *elastodynamics* is then relevant.

In order to formulate the governing equations of a continuous system we will resort to the theory of continuum mechanics where the equations of motion are expressed in terms of displacement field together with the boundary conditions to be satisfied. The space coordinates x, y, z being continuous, the system so described possesses an infinity of degrees of freedom.

In many cases, the specific geometry of the continuous bodies under investigation allows a simplified formulation of the equations of motion in terms of one or two displacement components, themselves functions of one or two space variables and time. Such situations, which are often encountered in practice, will be treated after a general presentation of the dynamic of continuous has been made.

3.3.1 The longitudinal vibrations of bar

A simple element in technical practice is a bar. His longitudinal dimension is much more greater as the transverse size. The next derivation are made on the bases of following suppositions:

1. The bar is axial symmetric
2. Sections perpendicular to the axis remain plane and perpendicular to the axis after the deformation

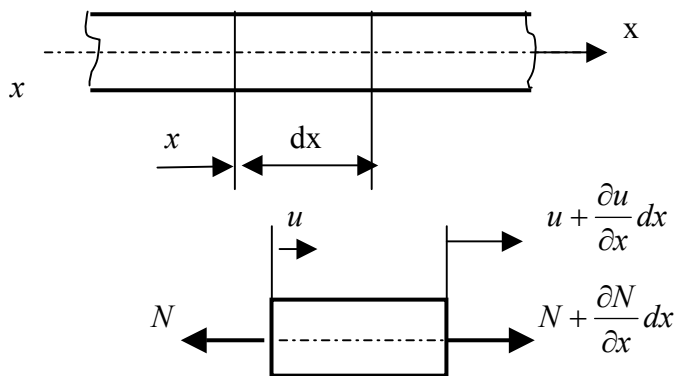


Fig. 3.11

3. The transverse deformations are neglected

From the bar we take out an element of length dx (Fig. 3.11)

We write the equation of motion of the element

$$N + \frac{\partial N}{\partial x} dx - N = \rho A dx \frac{\partial^2 u(x, t)}{\partial t^2} \quad (3.3.1)$$

The force in the bar is proportional to the strain

$$N = \sigma A = E \varepsilon A = E \frac{\partial u(x, t)}{\partial x} A$$

E is the *modulus of elasticity in tension (Yang modulus)*, ρ is the density of the material of the bar, A is the *cross section area* of the bar.

If we express N the partial differential equation of motion is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (3.3.2)$$

We used the notation

$$c = \sqrt{\frac{E}{\rho}} \quad (3.3.3)$$

c represents the *velocity of longitudinal waves propagation in the bar* or by other words it is the *velocity of sound in the bar*.

The partial differential equation we transform on an ordinary differential equation introducing the product of two functions – one is a function of a position and the other one is a function of time

$$u(x,t) = U(x)T(t)$$

Substituting this equation in (3.3.2) the partial differential equation changes in the ordinary one:

$$\frac{d^2 T}{dt^2} U = c^2 \frac{d^2 U}{dx^2} T$$

We separate the variables and put each side to the same value

$$\frac{1}{T} \frac{d^2 T}{dt^2} = c^2 \frac{1}{U} \frac{d^2 U}{dx^2} = -\Omega_0^2$$

Two equations will be obtained

$$\frac{d^2 T}{dt^2} + \Omega_0^2 T = 0 \quad a \quad \frac{d^2 U}{dx^2} + \frac{\Omega_0^2}{c^2} U = 0$$

Both equations represents the harmonic solution

$$T(t) = D_1 \cos \Omega_0 t + D_2 \sin \Omega_0 t \quad (3.3.4)$$

$$U(x) = C_1 \cos px + C_2 \sin px \quad (3.3.5)$$

$$p = \frac{\Omega_0}{c} = \Omega_0 \sqrt{\frac{\rho}{E}} \quad resp. \quad \Omega_0 = p \sqrt{\frac{E}{\rho}} \quad (3.3.6)$$

Integration constants C_1, C_2 will be obtained from *boundary conditions* and constants D_1, D_2 from *initial conditions*. The next solution depends on the arrangement of the bar.

The bar built in both sides (Fig. 3.12)

The boundary conditions are: for $x = 0$ $U(0) = 0$. Substituting in (3.3.5) we get $C_1 = 0$.

On the other side $x = l$ is $U(l) = 0$ and from (3.3.5) will be

$$C_2 \sin pl = 0$$

To fulfil this equation must be valid

$$pl = n\pi \text{ pro } n = 1, 2, \dots, \infty$$

Substituting this expression we obtain from (3.3.6) the circular natural frequency

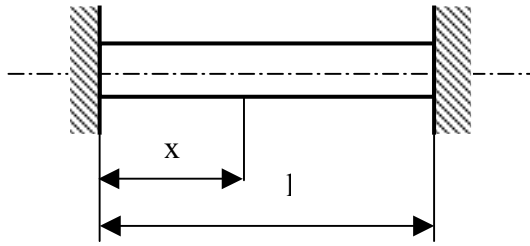


Fig. 3.12

$$\Omega_{0n} = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}} \quad (3.3.7)$$

The deformation of the bar will be given by the formula

$$u(x, t) = \sum_{n=1}^{\infty} (A \cos \Omega_{0n} t + B \sin \Omega_{0n} t) \sin \frac{n\pi}{l} x$$

The bar on one end built in and on the other end free (Fig. 3.13)

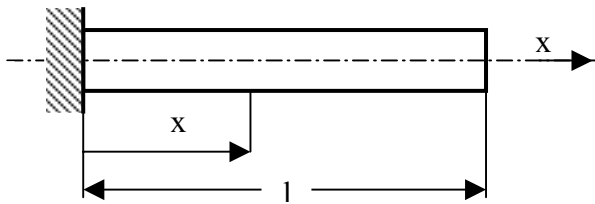


Fig 3 13

At the built in end the boundary condition gives $x = 0 \rightarrow U(0) = 0$. From this condition we get $C_1 = 0$;

At the free end the strain is equal to zero:

$$\frac{dU(l)}{dx} = C_{2p} \cos pl = 0$$

In this case it must be for non-trivial solution ($C_2 \neq 0$)

$$pl = \frac{\pi}{2}, \frac{3}{2}\pi, \dots, (2n-1)\frac{\pi}{2}, \dots$$

Substituting in (3.3.6) we get

$$\Omega_{0n} = \frac{(2n-1)\pi}{2l} \sqrt{\frac{E}{\rho}} \quad (3.3.8)$$

The bar on both sides free

In this case the boundary condition is

$$x=0 \rightarrow \frac{dU(0)}{dx} = 0 \quad \text{odkud} \quad C_2 = 0$$

$$\text{For } x=l \rightarrow \frac{dU(l)}{dx} = -C_1 p \cos pl = 0$$

For non-trivial solution is

$$pl = n\pi \quad \text{for } n = 1, 2, \dots, \infty$$

The natural circular frequency will be

$$\Omega_{0n} = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}} \quad (3.3.9)$$

The bar on one side built in and on the other side loaded by point mass (Fig.3.14)

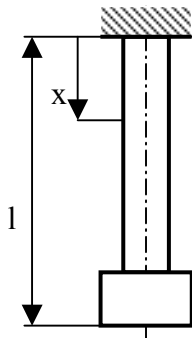


Fig. 3.14

The boundary condition in the built in end is

$$x=0 \rightarrow U(0) = 0 \rightarrow C_1 = 0$$

The equation of motion has the form

$$u(x,t) = C_2 \sin px(D_1 \cos \Omega_0 t + D_2 \sin \Omega_0 t)$$

The force on the free end must be

$$-N = m \frac{\partial^2 u(x,t)}{\partial t^2} = -EA \frac{\partial u(x,t)}{\partial x} = -EAC_2 p \cos px(D_1 \cos \Omega_0 t + D_2 \sin \Omega_0 t)$$

Using equation of motion it will be

$$EAp \cos pl = m\Omega_0^2 \sin pl$$

From here we get

$$\cotg pl = \frac{m\Omega_0^2}{EAp} = \frac{m}{m_p} pl$$

For $\frac{m}{m_p} = 1$ we can obtain $pl = 1,456\pi, 3\pi, 4,5\pi, \dots, \frac{3n}{2}\pi, \dots$ and then the natural circular

frequency

$$\Omega_{0n} = \frac{3\pi n}{2l} \sqrt{\frac{E}{\rho}}$$

Forced vibrations of a bar

We suppose a bar on one end built in and on the other side loaded by harmonic force (Fig.3.15)

At the fixed end of the bar the boundary condition is

$$x = 0 \quad U(0) = 0 \rightarrow C_1 = 0$$

The equation of motion will be

$$u(x,t) = C_2 \sin px \sin \omega t$$

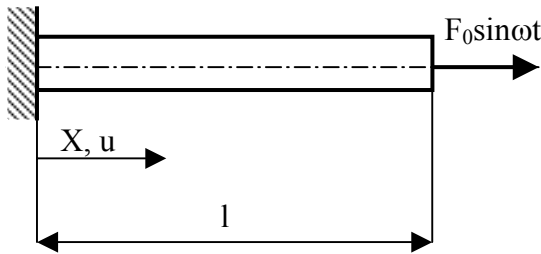


Fig. 3.15

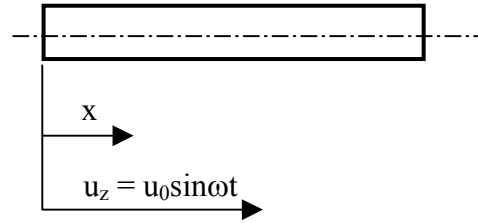


Fig. 3.16

At the free end the boundary condition is

$$x = l \quad EA \frac{\partial u(x,t)}{\partial x} = F_0 \sin \omega t$$

After substituting

$$EAC_2 p \cos pl = F_0$$

From this equation it will be determined the integration constant

$$C_2 = \frac{F_0}{EA p \cos pl}$$

Equation of motion of general section of the bar is

$$u(x,t) = \frac{F_0}{EA p \cos pl} \sin px \sin \omega t$$

After substituting $p = \omega / c$ we get

$$u(x,t) = \frac{F_0}{EA \frac{\omega}{c} \cos \frac{\omega}{c} l} \sin \frac{\omega}{c} x \sin \omega t \quad (3.3.10)$$

The resonance arises if the denominator of (3.3.10) will be zero, also for

$$\frac{\omega}{c}l = \frac{\pi}{2}, \frac{3}{2}\pi, \dots, (2n-1)\frac{\pi}{2}, \dots$$

The displacement of the free end is given by the equation

$$u(l,t) = \frac{F_0}{EA \frac{\omega}{c}} \operatorname{tg} \frac{\omega}{c} l \sin \omega t$$

The free bar excited kinematically

We consider the bar at the Fig. 3.16. His left side is excited kinematically by a harmonic function $u_z = u_0 \sin \omega t$. The solution is considered by the equation

$$u(x,t) = (C_1 \cos px + C_2 \sin px) \sin \omega t$$

At the left hand side is the boundary condition

$$x = 0 \rightarrow C_1 \sin \omega t = u_0 \sin \omega t \rightarrow C_1 = u_0$$

At the right hand side is does not act a force, therefore the stress and also the strain are equal to zero:

$$\left(\frac{\partial u(x,t)}{\partial x} \right)_{x=l} = u_0 l \sin pl + C_2 p \cos pl = 0$$

From here we obtain

$$C_2 = u_0 \operatorname{tg} pl$$

The equation of motion of a bar will be

$$u(x,t) = (u_0 \cos px + u_0 \operatorname{tg} pl \sin px) \sin \omega t$$

This equation is possible form on the following expression

$$u(x,t) = \frac{\cos \frac{\omega}{c} l \left(1 - \frac{x}{l} \right)}{\cos \frac{\omega}{c} l} u_0 \sin \omega t$$

From this shape is good seen the condition for resonance - $\cos \frac{\omega}{c} l = 0$ and so

$$\frac{\omega}{c}l = \frac{\pi}{2}, \frac{3}{2}\pi, \dots, (2n-1)\frac{\pi}{2}$$

3.3.2 Torque vibrations of shafts

A shaft of circular cross section is a very used element in machinery. By solution of his vibration we use the linear theory of elasticity. We consider prismatic shaft of circular cross section from which we cut out an element (3.17) and write its equation of motion

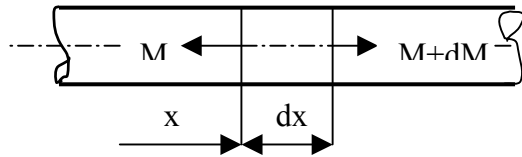


Fig. 3.17

$$\frac{\partial M}{\partial x} = \rho J_p \frac{\partial^2 \varphi(x, t)}{\partial t^2}$$

The torque is given by known expression from Strength of material

$$M = GJ_p \frac{\partial \varphi}{\partial x}$$

G is the modulus of shear, J_p is the quadratic polar moment of a cross section of the shaft. Using the expression for torque we get the equation of motion

$$c_t^2 \frac{\partial^2 \varphi(x, t)}{\partial x^2} = \frac{\partial^2 \varphi(x, t)}{\partial t^2} \quad (3.3.11)$$

We denoted the *velocity of shear waves propagation in the shaft*:

$$c_t = \sqrt{\frac{G}{\rho}} \quad (3.3.12)$$

Equation of motion (3.3.11) is similar to (3.3.2). Therefore the solution of the equation will be similar:

$$\varphi(x, t) = (\Phi_1 \cos px + \Phi_2 \sin px)(\Psi_1 \cos \Omega t + \Psi_2 \sin \Omega t) \quad (3.3.13)$$

In this expression we used

$$p = \frac{\Omega}{c_t} = \Omega \sqrt{\frac{\rho}{G}} \quad (3.3.14)$$

The determination of integration constants Φ_1 , Φ_2 , Ψ_1 a Ψ_2 will be provided from boundary and initial conditions.

3.3.3 Bending vibrations of beams

Beams are such elements of a structure, which can be loaded not only by axial forces but also by transversal load.

By derivation of equations of motion we suppose that

- The beam is straight
- Transverse deformations are small
- The vibration is in the plane given by an axis of the beam and one of the principle axis of quadratic moment of the cross section
- The planes perpendicular to the longitudinal axis of the beam remain plane and perpendicular to the axis by vibration
- Small displacements of shaft elements in the longitudinal direction are neglected

By the derivation of equations of motion we use Fig.3.18

The element makes a general plane motion. It is necessary to write the equation of motion of the translation

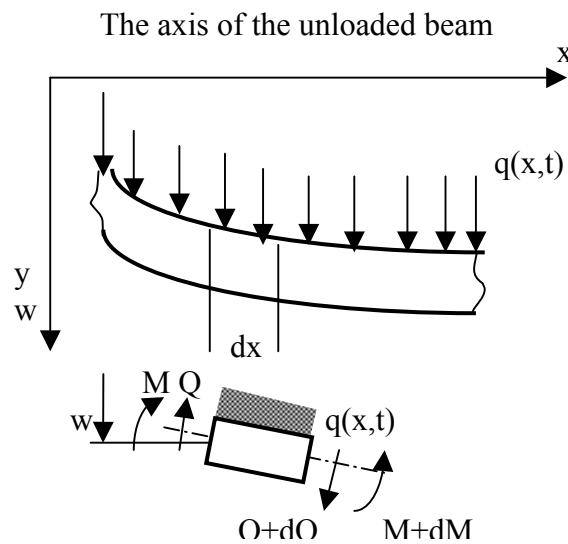


Fig. 3.18

$$Q + \frac{\partial Q}{\partial x} dx - Q + q(x,t)dx = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} dx$$

From here we get

$$\frac{\partial Q}{\partial x} + q(x,t) = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} \quad (3.3.15)$$

The equation of motion of rotation of the element about the mass centre is

$$Q - \frac{\partial M}{\partial x} = I_z \frac{\partial^2 \psi}{\partial t^2} \quad (3.3.16)$$

In these equation is ρ the density of the material of the beam, A the cross section area, Q the shearing force, M the bending moment, $w(x,t)$ the deflection of the beam, I_z the moment of inertia to the z – axis going through the mass centre, external continuous load on the unit length of the beam.

The rotation of the element is composed of the rotation due to the shear and bending

$$\frac{\partial w(x,t)}{\partial x} = \psi + \gamma \quad (3.3.17)$$

ψ is the rotation of the element (the slope of the deformed axis of the beam) given by known expression

$$EJ_z \frac{\partial \psi}{\partial x} = -M \quad (3.3.18)$$

J_z is the quadratic moment of the cross section to the axis z . γ is the shear of the cross section

$$\gamma = \frac{\kappa}{AG} Q \quad (3.3.19)$$

κ is the coefficient of shear deformation. From (3.3.17) we obtain

$$\psi = \frac{\partial w}{\partial x} - \gamma$$

After the first derivative with respect to x

$$\frac{\partial \psi}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial \gamma}{\partial x}$$

The bending moment from (3.3.18) will be

$$M = -EJ \frac{\partial^2 w}{\partial x^2} - EJ \frac{\partial \gamma}{\partial x}$$

From (3.3.16) we get

$$Q = I_z \frac{\partial^2 \psi}{\partial t^2} - EJ_z \frac{\partial^3 w}{\partial x^3} + EJ_z \frac{\partial^2 \gamma}{\partial x^2} = I_z \frac{\partial^3 w}{\partial x \partial t^2} - I_z \frac{\partial^2 \gamma}{\partial t^2} - EJ_z \frac{\partial^3 w}{\partial x^3} + EJ_z \frac{\partial^2 \gamma}{\partial x^2}$$

Substituting this equation in (3.3.15) and using the equation

$$\frac{\partial \gamma}{\partial x} = \frac{\kappa}{AG} \frac{\partial Q}{\partial x} = \frac{\kappa}{AG} \rho A \frac{\partial^2 w}{\partial t^2}$$

we obtain the equation of motion of the vibrating beam

$$\rho A \frac{\partial^2 w}{\partial t^2} + EJ \frac{\partial^4 w}{\partial x^4} - \underbrace{I_z \frac{\partial^4 w}{\partial x^2 \partial t^2}}_{\text{effect of rotation inertia}} + \underbrace{I_z \frac{\kappa \rho}{G} \frac{\partial^4 w}{\partial t^4} - EJ_z \frac{\kappa \rho}{G} \frac{\partial^4 w}{\partial x^2 \partial t^2}}_{\text{effect of shear}} = q(x, t) \quad (3.3.20)$$

While the moment of inertia of the element $I_z = (1/12)A\rho dx^3$ is essentially then the other terms it may be neglected. The effect of shear is by ordinary beams very small and is also neglected. The equation (3.3.20) will be much more simple

$$\rho A \frac{\partial^2 w(x, t)}{\partial t^2} + EJ \frac{\partial^4 w(x, t)}{\partial x^4} = q(x, t) \quad (3.3.21)$$

For abbreviation of writing we will not use by moment of inertia and quadratic moment of the cross section the index z .

3.3.3.1 The free vibrations of prismatic beam

In this chapter we consider the external load $q(x, t) = 0$.

he partial differential equation (3.3.21) we transfer to ordinary one by using

$$w(x, t) = W(x)T(t) \quad (3.3.22)$$

Then we use the notation

$$\frac{1}{\eta^4} = \frac{J E}{A \rho} = c_o^2 \frac{J}{A}$$

$c_o = \sqrt{\frac{E}{\rho}}$ is the velocity of bending wave propagation in the beam. After separation of

variables (3.3.21) becomes the form

$$\frac{1}{T(t)} \frac{d^2 T(t)}{dt^2} = \frac{1}{\eta^4 W(x)} \frac{d^4 W(x)}{dx^4} = -\Omega^2 \quad (3.3.23)$$

From here

$$\frac{d^2 T(t)}{dt^2} + \Omega^2 T(t) = 0$$

and its solution gives

$$T(t) = A_1 \cos \Omega t + A_2 \sin \Omega t \quad (3.3.24)$$

The next equation obtained from (3.3.23) is

$$\frac{d^4 W(x)}{dx^4} - \eta^4 \Omega^2 W(x) = 0$$

We introduce

$$p^4 = \eta^4 \Omega^2 \quad \text{nebo} \quad p = \eta \sqrt{\Omega} = \sqrt[4]{\frac{\rho A}{EJ} \Omega^2} \quad (3.3.25)$$

and so

$$\frac{d^4 W(x)}{dx^4} - p^4 W(x) = 0 \quad (3.3.26)$$

The solution of this equation is supposed in form

$$W(x) = Be^{\lambda x}$$

When substituting the solution in (3.3.26) we obtain the characteristic equation

$$\lambda^4 - p^4 = 0$$

The roots of this equation are

$$\lambda_1 = p; \quad \lambda_2 = -p; \quad \lambda_3 = ip; \quad \lambda_4 = -ip$$

Using these roots we get the equation of motion

$$W(x) = B_1 \sinh px + B_2 \cosh px + B_3 \sin px + B_4 \cos px \quad (3.3.27)$$

Substituting in (3.3.22) the equations (3.3.24) and (3.3.27)

$$w(x,t) = (B_1 \sinh px + B_2 \cosh px + B_3 \sin px + B_4 \cos px)(A_1 \cos \Omega t + A_2 \sin \Omega t) \quad (3.3.28)$$

The general solution will be obtained by linear combination of all modes

$$w(x,t) = \sum_{i=1}^{\infty} (B_{1i} \cosh p_i x + B_{2i} \sinh p_i x + B_{3i} \sin p_i x + B_{4i} \cos p_i x)(A_{1i} \cos \Omega_i t + A_{2i} \sin \Omega_i t) \quad (3.3.29)$$

The integration constants B_1, B_2, B_3, B_4 which determines the shape of the vibrating beam we get from boundary conditions and the integration constants A_1 and A_2 from initial conditions. In order to be liable to determine the integration constants *Krylov function are used*, sometimes called *Rayleigh functions*. These function are developed so, that one of this function is for zero equal to one and the other functions vanish. These form of its is following:

$$\begin{aligned} S(px) &= \frac{1}{2}(\cosh px + \cos px) \\ T(px) &= \frac{1}{2}(\sinh px + \sin px) \\ U(px) &= \frac{1}{2}(\cosh px - \cos px) \\ V(px) &= \frac{1}{2}(\sinh px - \sin px) \end{aligned} \quad (3.3.30)$$

The next property is that its derivatives is possible obtain by cyclic exchange of the previous function

$$\begin{aligned}
S'(px) &= pV(px); S''(px) = p^2U(px); S'''(px) = p^3T(px) \\
T'(px) &= pS(px); T''(px) = p^2V(px); T'''(px) = p^3U(px) \\
U'(px) &= pT(px); U''(px) = p^2S(px); U'''(px) = p^3V(px) \\
V'(px) &= pU(px); V''(px) = p^2T(px); V'''(px) = p^3S(px)
\end{aligned} \tag{3.3.31}$$

From (3.3.30) we see that for $x = 0$ is $S(0) = 1$ and $T(0) = U(0) = V(0) = 0$. Using Krylov functions (3.3.27) we can write in the form

$$W(x) = B_1S(px) + B_2T(px) + B_3U(px) + B_4V(px) \tag{3.3.32}$$

In these expressions they occurs hyperbolic functions, which reach by greater arguments very large values. Therefore is introduced instead x the dimensionless value $\xi = x/l$. If l is the total length of the beam, ξ is in interval $\xi \in (0,1)$. When we use ξ some previous expressions will be changed

$$\frac{dW(x)}{dx} = \frac{dW(\xi)}{d\xi} \frac{d\xi}{dx} = \frac{dW(\xi)}{d\xi} \frac{1}{l}$$

Similar

$$\frac{d^4W(x)}{dx^4} = \frac{d^4W(\xi)}{d\xi^4} \frac{1}{l^4}$$

The equation (3.3.26) now is

$$\frac{d^4W(\xi)}{d\xi^4} - p^4l^4W(\xi) = 0$$

For next solutions we introduce

$$\lambda = pl = l\sqrt{\frac{\rho A}{EJ}}\Omega^2 \tag{3.3.33}$$

$$\Omega = \frac{\lambda^2}{l^2} \sqrt{\frac{EJ}{\rho A}} \tag{3.3.34}$$

The equation of motion is now

$$\frac{d^4W(\xi)}{d\xi^4} - \lambda^4W(\xi) = 0$$

Its solution is

$$W(\xi) = B_1S(\lambda\xi) + B_2T(\lambda\xi) + B_3U(\lambda\xi) + B_4V(\lambda\xi) \tag{3.3.35}$$

Besides this equation we well need for the next solutions

the slope of deflection curve

$$\frac{dW(\xi)}{d\xi} = \frac{\lambda}{l} [B_1V(\lambda\xi) + B_2S(\lambda\xi) + B_3T(\lambda\xi) + B_4(\lambda\xi)] \quad (3.3.36)$$

The equation of bending moment

$$\bar{M}(\xi) = \frac{M(\xi)}{T(\xi)} = -\frac{EJ\lambda^2}{l^2} [B_1U(\lambda\xi) + B_2V(\lambda\xi) + B_3S(\lambda\xi) + B_4T(\lambda\xi)] \quad (3.3.37)$$

The equation of shearing force

$$\bar{Q}(\xi) = \frac{Q(\xi)}{T(\xi)} = -\frac{EJ\lambda^3}{l^3} [B_1T(\lambda\xi) + B_2U(\lambda\xi) + B_3V(\lambda\xi) + B_4S(\lambda\xi)] \quad (3.3.38)$$

As an example we will solve the cantiliver beam shown in Fig.3.19.

At the built in side the boundary conditions are $x = \xi = 0 \rightarrow W(0) = 0$ and $W'(0)=0$.

From (3.3.35) is $B_1 = 0$ and from ((3.3.36) $B_2 = 0$. From (3.3.37) and (3.3.38) we get

$$\begin{aligned} B_3S(\lambda) + B_4T(\lambda) &= 0 \\ B_3V(\lambda) + B_4S(\lambda) &= 0 \end{aligned}$$

We received the system of two homogenous equations with unknown constants $B_3 B_4$. For non-trivial solution must be the determinant of the system be zero:

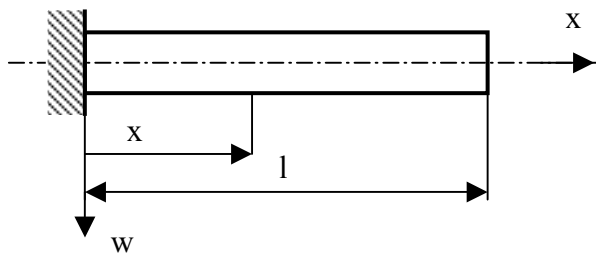


Fig.3.19

$$\begin{vmatrix} S(\lambda) & T(\lambda) \\ V(\lambda) & S(\lambda) \end{vmatrix} = S^2(\lambda) - T(\lambda)V(\lambda) = 0$$

Substituting for $S(\lambda)$, $T(\lambda)$ a $V(\lambda)$ from (3.3.31) is obtained

$$1 + \cos \lambda \cosh \lambda = 0$$

After numerical solution of this equation¹⁾ we obtain the roots

$$\lambda_1 = 0,5968\pi; \quad \lambda_2 = 1,4942\pi; \quad \lambda_3 = 2,5003\pi; \quad \lambda_n = (n - \frac{1}{2})\pi$$

When we use these roots we determine from (3.3.34) the natural frequencies. Using some of λ_n it is possible get the mode of vibrating beam

¹⁾ It is possible to use the method of half step, method of secants, Newton's method or mathematical sw MAPLE or MATLAB

$$\frac{B_4}{B_3} = -\frac{S(\lambda_n)}{T(\lambda_n)} = -\frac{\cosh \lambda_n + \cos \lambda_n}{\sinh \lambda_n + \sin \lambda_n}$$

When we use $\lambda_n = (n - \frac{1}{2})\pi$ we

$$\bar{W}_n(\xi) = U(\lambda_n \xi) - \frac{\cosh \lambda_n + \cos \lambda_n}{\sinh \lambda_n + \sin \lambda_n} V(\lambda_n \xi)$$

The shape of the modes is shown on Fig. 3.20

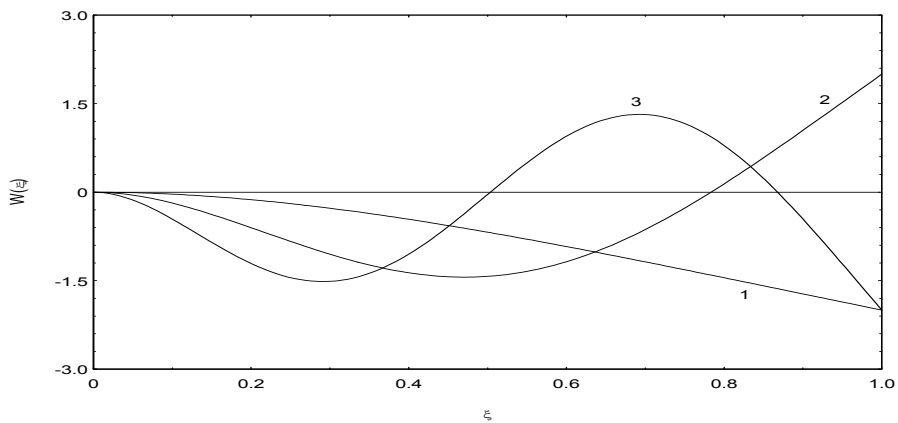
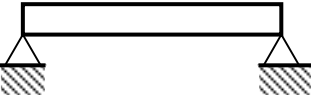
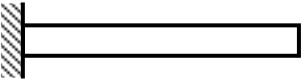
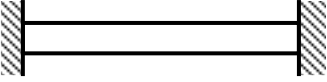
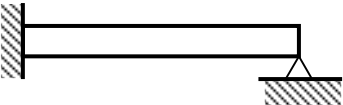



Fig. 3.20

For different modes there exists places with zero displacement during the all time of vibration. These places are called *nodal point*. Its number is $N = n - 1$, where n is the number of mode. By similar way it is possible to solve other types of beams. The results are in Tab.3.3.1

Tab. 3.3.1 The solution of basic types of beams.

Type of beam boundary conditions	Characteristic equation	Roots of characteristic equation
 $W(0) = 0$ $W(l) = 0$ $M(0) = 0$ $M(l) = 0$	$\sin \lambda - \sinh \lambda = 0$	$\lambda_n = n\pi$
 $W(0) = 0$ $M(l) = 0$ $W'(0) = 0$ $Q(l) = 0$	$1 + \cos \lambda - \cosh \lambda = 0$	$\lambda_1 = 0,5968\pi; \lambda_2 = 1,4942\pi$ $\lambda_n = (n - \frac{1}{2})\pi$
 $W(0) = 0$ $W(l) = 0$ $W'(0) = 0$ $W'(l) = 0$	$\cos \lambda \cosh \lambda - 1 = 0$	$\lambda_n = (n + \frac{1}{2})\pi$
 $W(0) = 0$ $W(l) = 0$ $W'(0) = 0$ $M(l) = 0$	$tg \lambda - tgh \lambda = 0$	$\lambda_n = (n + \frac{1}{4})\pi$
 $M(0) = 0$ $M(l) = 0$ $Q(0) = 0$ $Q(l) = 0$	$\cos \lambda \cosh \lambda - 1 = 0$	$\lambda_n = (n + \frac{1}{2})\pi$
$\Omega_0 = \frac{\lambda_n^2}{l^2} \sqrt{\frac{EJ}{\rho A}}$		

3.3.3.2 Bending vibration of beam excited by concentrated load

Let us consider excitation by a concentrated force F acting at the distance x_1 . When we want to use the equation (3.3.21) we consider the force continuously distributed on the length Δ (Fig.3.21). The force is supposed as a harmonic function of time $F = F_1 \sin \omega t$.

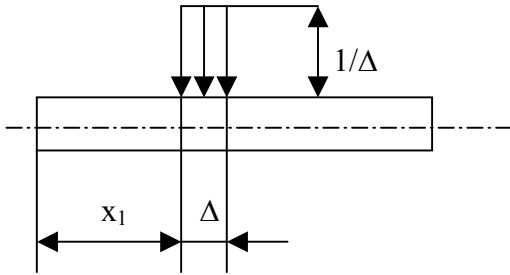


Fig. 3.21

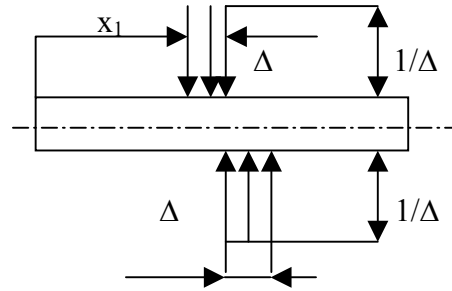


Fig. 3.22

Equation of motion will be

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EJ \frac{\partial^4 w(x,t)}{\partial x^4} = \delta_1(x) F_1 \sin \omega t \quad (3.3.39)$$

$\delta_1(x)$ is the Dirac function with following values

$$\begin{aligned} \delta_1(x) &= \frac{1}{\Delta} \quad \text{for } x_1 \leq x \leq x_1 + \Delta \\ \delta_1(x) &= 0 \quad \text{for } x < x_1 \quad \text{and} \quad x > x_1 + \Delta \end{aligned}$$

This expression has the advantage that (3.3.39) is valid in the whole length of the beam. The partial differential equation will be transferred on the ordinary one using the arrangement

$$w(x,t) = W(x) \sin \omega t$$

Then will be

$$\frac{d^4 W(x)}{dx^4} - \frac{\omega^2 \rho A}{EJ} W(x) = \frac{F_1 \delta_1(x)}{EJ}$$

We use the notation

$$p = \sqrt[4]{\frac{\rho A}{EJ}} \omega^2$$

The previous equation can we write in the form

$$\frac{d^4 W(x)}{dx^4} - p^4 W(x) = \frac{F_1 \delta_1(x)}{EJ} \quad (3.3.40)$$

We got ordinary differential equation of 4th order with constant coefficients. Its solution consists of the homogenous solution and a particular one:

$$W(x) = B_1 S(px) + B_2 T(px) + B_3 U(px) + B_4 V(px) + \Phi(x) \quad (3.3.41)$$

The particular solution will be determined, by using Duhamel integral

$$\Phi(x) = \frac{F_1}{p^3 EJ} \int_0^x \delta_1(\bar{x}) V[p(x - \bar{x})] d\bar{x}$$

The determination of $\Phi(x)$ is reasonable only in the interval $x_1 \leq \bar{x} \leq x_1 + \Delta$. In such case is

$$\Phi(x) = \frac{F_1}{p^3 EJ} \int_{x_1}^{x_1 + \Delta} \frac{1}{\Delta} V[p(x - \bar{x})] d\bar{x} = \frac{F_1}{p^3 EJ} V[p(x - x_1)]$$

In the case when the beam is loaded by external bending moment it is possible to replace the moment by a couple and each of these distributed forces by its resultant. The Dirac function obtains the shape

$$\begin{aligned} \delta_2(x) &= \frac{1}{\Delta^2} \quad \text{pro } x_1 \leq x \leq x_1 + \Delta \\ \delta_2(x) &= -\frac{1}{\Delta^2} \quad \text{pro } x_1 + \Delta \leq x \leq x_1 + 2\Delta \\ \delta_2(x) &= 0 \quad \text{pro } x < x_1 \quad \text{a } x > x_1 + 2\Delta \end{aligned}$$

Particular solution will express the equation

$$\Phi(x) = \frac{M_1}{p^2 EJ} U[p(x - x_1)]$$

(3.3.41) obtains the form

$$W(x) = B_1 S(px) + B_2 T(px) + B_3 U(px) + B_4 V(px) + \frac{F_1}{p^3 EJ} V[p(x - x_1)] + \frac{M_1}{p^2 EJ} U[p(x - x_1)]$$

(3.3.42)

The last two members of the right hand side enter into the determination of the deflection of the beam behind the place of action of external load. If on the beam the point mass is placed then the force F_1 will be replaced by the expression

$$F_1 = m\Omega^2 W(x_1) \quad (3.3.43)$$

When we use the dimensionless parameter $\xi=x/l$ (3.3.43) gets the form

$$W(\xi) = B_1 S(\lambda\xi) + B_2 T(\lambda\xi) + B_3(\lambda\xi) + B_4(\lambda\xi) + \frac{F_1}{\lambda^3 EJ} V[\lambda(\xi - \xi_1)] + \frac{M_1}{\lambda^2 EJ} U[\lambda(\xi - \xi_1)] \quad (3.3.44)$$

$$\lambda = pl$$

3.3.3.3 The method of transfer matrices

This method stay on the fact, that all needed parameters of the beam is possible to determine from the parameters on the beginning of the beam ($x = 0$ or $\xi = 0$). From equations (3.3.35),

(3.3.36), (3.3.37) and (3.3.38) is possible to determine the integration constants:

$$B_1 = W(0); B_2 = \frac{l}{\lambda} W'(0); B_3 = -\frac{l^2}{\lambda^2} \frac{M(0)}{EJ}; B_4 = -\frac{l^3}{\lambda^3} \frac{Q(0)}{EJ}$$

By using of these integration constants we determine the parameters in the section of the beam

$$\begin{aligned}
W(\xi) &= W(0)S(\lambda\xi) + \frac{W'(0)l}{\lambda}T(\lambda\xi) - \frac{M(0)l^2}{\lambda^2 EJ}U(\lambda\xi) - \frac{Q(0)l^3}{\lambda^3 EJ}V(\lambda\xi) + \\
&\quad + \frac{F_1 l^3}{\lambda^3}V[\lambda(\xi - \xi_1)] + \frac{M_1 l^2}{\lambda^2 EJ}U[\lambda(\xi - \xi_1)] \\
W'(\xi) &= \frac{\lambda}{l}W(0)V(\lambda\xi) + W'(0)S(\lambda\xi) - \frac{M(0)l}{\lambda EJ}T(\lambda\xi) - \frac{Q(0)l^2}{\lambda^2 EJ}U(\lambda\xi) + \\
&\quad + \frac{F_1 l^2}{\lambda^2 EJ}U[\lambda(\xi - \xi_1)] + \frac{M_1 l}{\lambda EJ}T[\lambda(\xi - \xi_1)] \\
-M(\xi) &= \frac{\lambda^2 EJ}{l^2}W(0)U(\lambda\xi) + \frac{\lambda EJ}{l}W'(0)V(\lambda\xi) - M(0)S(\lambda\xi) - \frac{Q(0)l}{\lambda}T(\lambda\xi) + \\
&\quad + \frac{F_1 l}{\lambda}T[\lambda(\xi - \xi_1)] + M_1 S[\lambda(\xi - \xi_1)] \\
-Q(\xi) &= \frac{\lambda^3 EJ}{l^3}W(0)T(\lambda\xi) + \frac{\lambda^2 EJ}{l^2}W'(0)U(\lambda\xi) - \frac{\lambda}{l}M(0)V(\lambda\xi) - Q(0)S(\lambda\xi) + \\
&\quad + F_1 S[\lambda(\xi - \xi_1)] + \frac{\lambda}{l}M_1 V[\lambda(\xi - \xi_1)]
\end{aligned} \tag{3.3.45}$$

The last two terms in each of the previous equations are equal to zero for $\xi \leq \xi_1$. The equations (3.3.45) is possible to write in matrix form.

We define the *status vector* at the beginning of i section

$$\mathbf{s}_i = [W, W', -M, -Q]_i^T$$

and the status vector on the end of the i section, which is also the status vector at the beginning of $i + 1$ section

$$\mathbf{s}_{i+1} = [W, W', -M, -Q]_{i+1}^T$$

We define the *transfer matrix* \mathbf{P}_i . By using of it we determine the status vector on the beginning of the $I + 1$ section

$$\mathbf{s}_{i+1} = \mathbf{P}_i \mathbf{s}_i \tag{3.3.46}$$

While the end of one section is ever the beginning point of the next section it is define the value on the end of $n + 1$ section

$$\mathbf{s}_{n+1} = \mathbf{P}_n \mathbf{P}_{n-1} \mathbf{P}_{n-2} \dots \mathbf{P}_0 \mathbf{s}_0 \tag{3.3.47}$$

or

$$\mathbf{s}_{n+1} = \mathbf{P}_{0,n} \mathbf{s}_0 \tag{3.3.48}$$

The resultant transfer matrix is given by the product of transfer matrices all sections

$$\mathbf{P}_{0,n} = \mathbf{P}_n \mathbf{P}_{n-1} \mathbf{P}_{n-2} \dots \mathbf{P}_0 = \prod_{i=n}^0 \mathbf{P}_i \quad (3.3.49)$$

The transfer matrix is square of order 4, because the status vector is also of order 4. By multiplication of transfer matrices arise always the square matrix of order 4. This is advantage of this method and the calculation is possible by using PC. The next advantage is in the fact, that we can easy built the computational model of a set of beams with any support , with variable stiffness and cross section as well as statically indeterminate. The only disadvantage is, that by multiplication of many transfer matrices it can occur the numerical instability.

By application, the set of beans is divided on sections, each with constant geometrical, mass, stiffness and force parameters. Any section may be arbitrary short and so define the section with concentrated force, concentrated moment, point mass, etc. We show the transfer marices for some types of beam sections.

Prismatic beam of length l (Fig. 3.23)

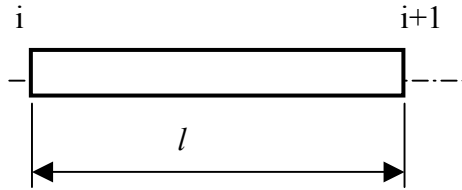


Fig. 3.23

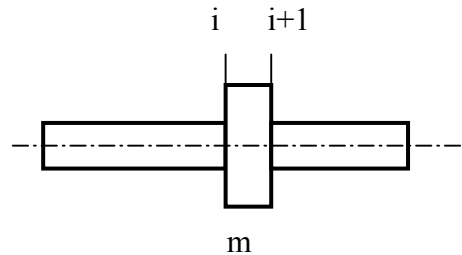


Fig. 3.24

$$\mathbf{P}_i = \begin{bmatrix} S(\lambda) & \frac{1}{\lambda}T(\lambda) & \frac{l^2}{\lambda^2 EJ}U(\lambda) & \frac{l^3}{\lambda^3 EJ}V(\lambda) \\ \frac{\lambda}{l}V(\lambda) & S(\lambda) & \frac{l}{\lambda EJ}T(\lambda) & \frac{l^2}{\lambda^2 EJ}U(\lambda) \\ \frac{\lambda^2}{l^2}EJU(\lambda) & \frac{\lambda}{l}EJV(\lambda) & S(\lambda) & \frac{l}{\lambda}T(\lambda) \\ \frac{\lambda^3}{l^3}EJT(\lambda) & \frac{\lambda^2}{l^2}EJU(\lambda) & \frac{\lambda}{l}V(\lambda) & S(\lambda) \end{bmatrix} \quad (3.3.50)$$

Rigid mass point (Fig.3.24)

The rigid mass point changes the shear force of the value $am = -\Omega^2 Wm$. The transfer matrix of this case will be

$$\mathbf{P}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\Omega^2 & 0 & 0 & 1 \end{bmatrix} \quad (3.3.51)$$

Rigid mass with moment of inertia

This case is similar to the case shown in Fig.3.24 but in position $i + 1$ will change also the moment of the value $-\Omega^2 W I$. The transfer matrix is

$$\mathbf{P}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \Omega^2 I & 1 & 0 \\ m\Omega^2 & 0 & 0 & 1 \end{bmatrix} \quad (3.3.52)$$

Elastic support with stiffness coefficient k (Fig. 3.25)

At the end of the section will be changes the shear force of the value kW

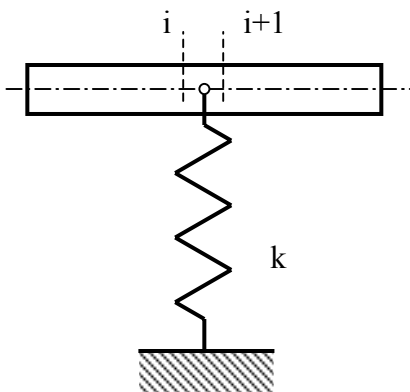


Fig. 3.25

$$\mathbf{P}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.3.52)$$

The elastic hinge with the stiffness coefficient κ

In the section will the bending moment of the value $-\kappa W'$:

$$\mathbf{P}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.3.53)$$

As example we solve a cantilever beam of variable cross section (Fig.3.26), which can be a model of turbine blade. We divide the beam on prismatic sections. The lengths l_i of each section may be different. The transfer matrix of each section is given by the equation (3.3.50). The length l_i , value λ_i , quadratic moment of the cross section J_i . Yang modulus E and density ρ are in all sections equal. The resultant transfer matrix will be determined from (3.3.49). We consider the boundary conditions: at fixed end $W(0) = 0$, $W'(0) = 0$. At the free end $M = 0$ a $Q = 0$. We will write symbolic

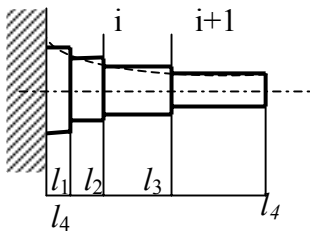


Fig.3.26

$$\begin{bmatrix} W_{4+1} \\ W'_{4+1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -M_0 \\ -Q_0 \end{bmatrix}$$

p_{ij} are elements of the resultant transfer matrix. The matrix notation represent 4 equations, which will have the form

$$\begin{aligned} W_{4+1} + 0 + 0 + p_{13}M_0 + p_{14}Q_0 &= 0 \\ 0 + W'_{4+1} + 0 + p_{23}M_0 + p_{24}Q_0 &= 0 \\ 0 + 0 + 0 + p_{33}M_0 + p_{34}Q_0 &= 0 \\ 0 + 0 + 0 + p_{43}M_0 + p_{44}Q_0 &= 0 \end{aligned}$$

The system of equations is homogenous. Therefore for the non-trivial solution must be the determinant of the system equal to zero: Because the 3rd column contain only 0 it must not be taken into account and the determinant will be

$$\begin{vmatrix} 1 & 0 & p_{13} & p_{14} \\ 0 & 1 & p_{23} & p_{24} \\ 0 & 0 & p_{33} & p_{34} \\ 0 & 0 & p_{43} & p_{44} \end{vmatrix} = 0$$

After evaluation of the determinant we get

$$p_{33}p_{44} - p_{34}p_{43} = 0$$

By the solution of this equation we get the natural circular frequency of the beam and after that the modes of vibrations.

3.3.3.4 The influence of rotational inertia and the shear

As far all solutions were provided by using of the simplified equation of motion (3.3.21). The differences between this solution and the exact solution given by (3.3.20) shows the Fig. 3.27. On the vertical axis is the ratio of the exact solution to the simplified solution. On the horizontal axis is the ratio of the length of the beam to the radius of the quadratic moment of the cross section $j = \sqrt{\frac{J}{A}}$.. From the figure is seen that greater differences occur by very short and high beams.

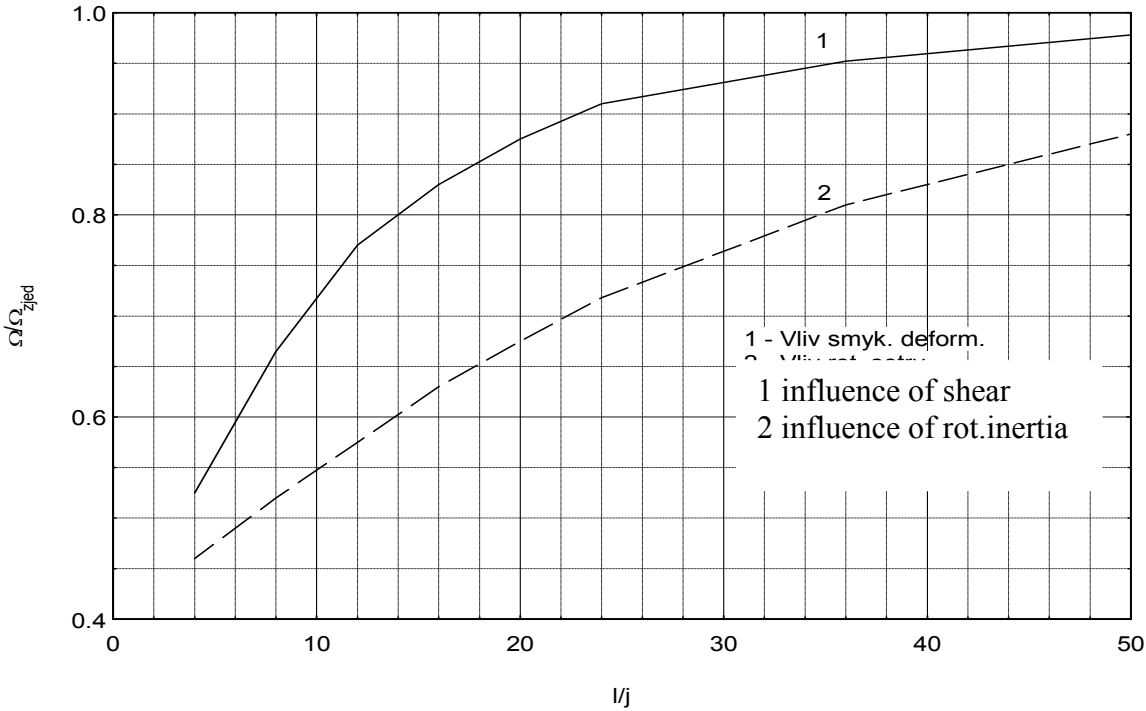


Fig. 3.27

3.3.4 Vibration of membranes

A membrane is a skin which is stretched with a tension and which has no bending stiffness whatever. It is considered as two-dimensional mechanical system.

Let us consider that the membrane is stretched in all directions by relative tension N (force per unit length). The relative mass of membrane is q (mass per unit surface). The Fig. 3.28 shows the cutting through an element of a membrane.

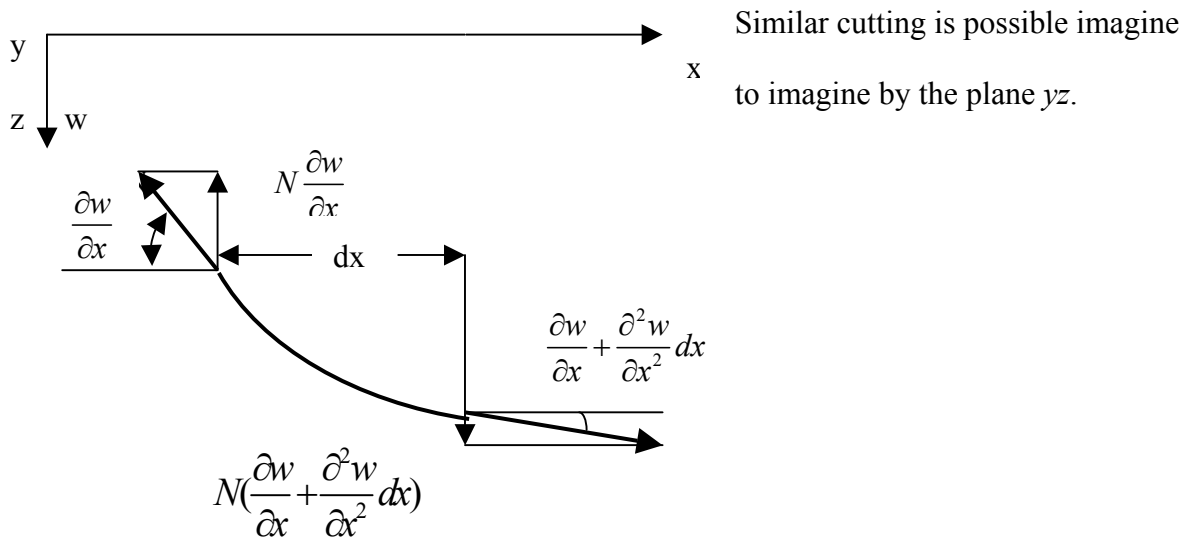


Fig. 3.28

We can write the equation of motion

$$N \left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right) dy + N \left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy \right) dx - N \left(\frac{\partial w}{\partial x} dy + \frac{\partial w}{\partial y} dx \right) = q dx dy \frac{\partial^2 w}{\partial t^2}$$

After arrangement of this equation we get

$$N \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = q \frac{\partial^2 w}{\partial t^2} \quad (3.3.54)$$

It is possible to use Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and we can write (3.3.54) in the form

$$N \nabla^2 w(x, y, t) - q \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0 \quad (3.3.55)$$

We suppose harmonic vibration

$$w(x, y, t) = W(x, y) \sin \Omega t$$

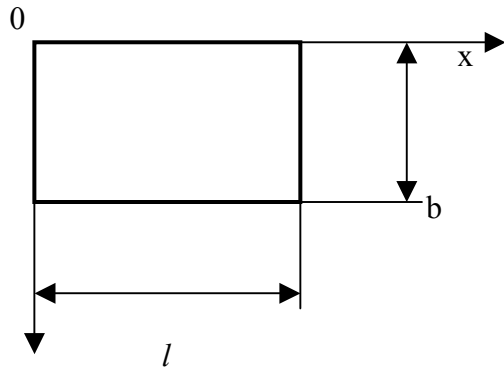
After substituting in (3.3.55)

$$q\Omega^2 W(x, y) + N\nabla^2 W(x, y) = 0 \quad (3.3.56)$$

The solution of this equation depends on the boundary conditions.

5.3.4.1 Rectangular membrane

Let us consider that the membrane is created by axes x, y and the straight lines parallel with



y

Fig . 3.29

them ($x = l, y = b$) Fig. 3.29

On the circumference of the membrane the deflection are zero

$$w(0, y) = 0; \quad w(x, 0) = 0; \quad w(l, y) = 0; \quad w(x, b) = 0$$

We chose the solution that suit to these

boundary conditions:

$$w_{i,j}(x, y) = C \sin \frac{i\pi x}{l} \sin \frac{j\pi y}{b} \quad (i, j = 1, 2, \dots, \infty)$$

The second derivatives of these equations with respect to x and y will be

$$\frac{\partial^2 w}{\partial x^2} = -\frac{i^2 \pi^2}{l^2} \sin \frac{i\pi x}{l} \sin \frac{j\pi y}{b} = -\frac{i^2 \pi^2}{l^2} w$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{j^2 \pi^2}{b^2} \sin \frac{i\pi x}{l} \sin \frac{j\pi y}{b} = -\frac{j^2 \pi^2}{b^2} w$$

After substituting in (3.3.56) we get

$$\left[q\Omega_{i,j}^2 - N \left(\frac{i^2 \pi^2}{l^2} + \frac{j^2 \pi^2}{b^2} \right) \right] w = 0$$

The circular natural frequencies are calculated from this equation

$$\Omega_{i,j}^2 = \frac{N}{q} \pi^2 \left(\frac{i^2}{l^2} + \frac{j^2}{b^2} \right) \text{ for } i, j = 1, 2, \dots, \infty \quad (3.3.57)$$

The general solution is

$$w(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin \frac{i\pi x}{l} \sin \frac{j\pi y}{b} (A_{i,j} \cos \Omega_{i,j} t + B_{i,j} \sin \Omega_{i,j} t) \quad (3.3.58)$$

or

$$w(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} C_{i,j} \sin \frac{i\pi x}{l} \sin \frac{j\pi y}{b} \sin(\Omega_{i,j} t + \varphi_{i,j}) \quad (3.3.59)$$

By vibration of membranes are important the *nodal lines* – the places in which the displacement are in any time equal to zero. From (3.3.58) or (3.3.59) that it will be if

$$x = \frac{l}{i}, \frac{2l}{i}, \dots, \frac{i-1}{i} l$$

$$y = \frac{b}{j}, \frac{2b}{j}, \dots, \frac{j-1}{j} b$$

The nodal lines divides the membrane $i*j$ same parts (Fig.3.30)

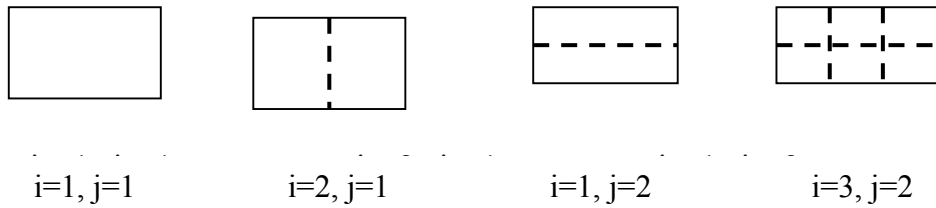


Fig. 3.30

The nodal lines have not to be only a lines parallel to sides of a membrane.

Let we consider a square membrane ($b = l$) and suppose the mode of vibration $i = 1, j = 2$.

(3.3.59) will be

$$w(x, t) = \left(A \sin\left(\frac{2\pi}{l} x\right) \sin\left(\frac{\pi}{l} y\right) + B \sin\left(\frac{\pi}{l} x\right) \sin\left(\frac{2\pi}{l} y\right) \right) \sin(\Omega_{ij} t + \varphi_{ij}) = 0$$

The nodal lines does not depend on time. After arrangement of this equation we get

$$\sin \frac{\pi}{l} x \sin \frac{\pi}{l} y \left(\cos \frac{\pi}{l} x + \lambda \cos \frac{\pi}{l} y \right) = 0$$

where is $\lambda = \frac{B}{A}$

This condition is fulfilled if

$$\sin \frac{\pi}{l} x = 0, \text{ resp. } \sin \frac{\pi}{l} y = 0$$

or

$$\cos \frac{\pi}{l} x + \lambda \cos \frac{\pi}{l} y = 0$$

The first condition gives the nodal lines parallel to sides of the membrane, the results of the second condition depend on λ . If $\lambda = -1$ the equation of the line is $y = x$ (Fig.3.31a), if $\lambda = 1$ the equation of the line is $y = -x$ (Fig.3.31b). If $\lambda = -2$ the shape of nodal line is given by the equation $\cos \frac{\pi x}{l} = 2 \cos \frac{\pi}{l} y$ (Fig.3.31c).

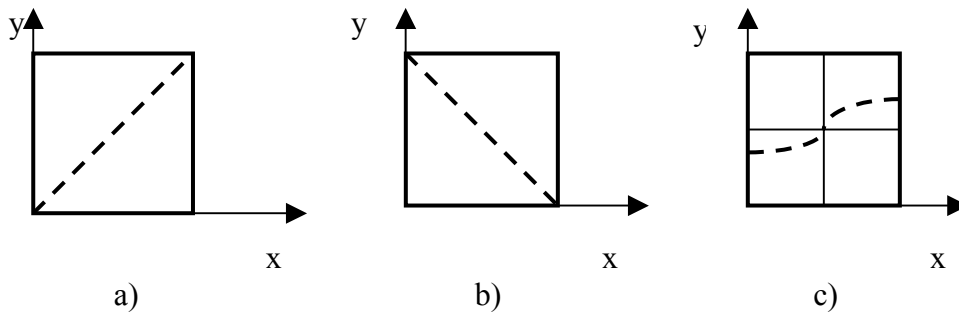


Fig 3 31

3.3.4.2 Circular membranes

We consider a circular membrane of radius R . It is advantageous to use polar coordinates

$$x = r \cos \vartheta$$

$$y = r \sin \vartheta$$

The Laplace operator has in polar coordinates the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2}$$

The equation of motion (3.3.55) in polar coordinates is

$$N \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} \right) w(r, \vartheta, t) - q \frac{\partial^2 w(r, \vartheta, t)}{\partial t^2} = 0 \quad (3.3.60)$$

We divide this equation by q and use

$$v = \sqrt{\frac{N}{q}}$$

(3.3.60) changes to

$$\frac{\partial^2 w(r, \vartheta, t)}{\partial t^2} = v^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} \right) w(r, \vartheta, t)$$

We transform this partial differential equation to an ordinary one using for deflection following expression

$$w(r, \vartheta, t) = \bar{r}(r) \bar{\vartheta}(\vartheta) \bar{t}(t)$$

The ordinary differential equation is

$$\frac{1}{t} \frac{d^2 \bar{t}}{dt^2} = \frac{v^2}{\bar{r}} \left(\frac{d^2 \bar{r}}{dr^2} + \frac{1}{r} \frac{d\bar{r}}{dr} \right) + \frac{v^2}{r^2 \bar{\vartheta}} \frac{d^2 \bar{\vartheta}}{d\vartheta^2} \quad (3.3.61)$$

Because the right hand side of the equation is independent must be also the left hand side time independent. We put both sides of (3.3.61) to $-\Omega_0^2$. Then it is

$$\frac{d^2 \bar{t}}{dt^2} + \Omega_0^2 \bar{t} = 0$$

We obtained the differential equation of harmonic motion

$$\bar{t} = C \cos \Omega_0 t + D \sin \Omega_0 t$$

In (3.3.61) must be for axial symmetrical membrane the term

$$\frac{1}{\bar{\vartheta}} \frac{d^2 \bar{\vartheta}}{d\vartheta^2} = -n^2$$

constant and by its solution we get

$$\bar{\vartheta} = A \cos n\vartheta + B \sin n\vartheta$$

Substituting this value in (3.3.61) we obtain

$$-\Omega_0^2 = \frac{v^2}{r} \left(\frac{d^2 \bar{r}}{dr^2} + \frac{1}{r} \frac{d\bar{r}}{dr} \right) - \frac{v^2 n^2}{r^2}$$

When we use

$$k = \frac{\Omega_0}{v}$$

$$r^2 \frac{d^2 \bar{r}}{dr^2} + r \frac{d\bar{r}}{dr} + (k^2 r^2 - n^2) \bar{r} = 0$$

This equation is known as *Bessel equation* and its solution is

$$\bar{r} = EJ_n(kr) + FY_n(kr)$$

Here E, F are constants and $J_n(kr)$ and $Y_n(kr)$ are Bessel function of first and second art.

The deformation of circular membrane is given by the expression

$$w = (A \cos n\vartheta + B \sin n\vartheta)(C \cos \Omega_0 t + D \sin \Omega_0 t)(EJ_n(kr) + FY_n(kr)) \quad (3.3.62)$$

While the Bessel function $Y_n(0) = \infty$ $F = 0$ and (3.3.62) obtains the form

$$w = (A \cos n\vartheta + B \sin n\vartheta)(C \cos \Omega_0 t + D \sin \Omega_0 t)EJ_n(kr) \quad (3.3.63)$$

If the vibration is symmetric to the axis of the membrane $n = 0$ and (3.3.63) is possible to simplifies

$$w = (C \cos \Omega_0 t + D \sin \Omega_0 t)J_n(kr)$$

The Bessel function has the value

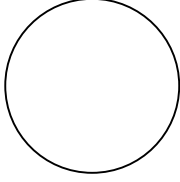
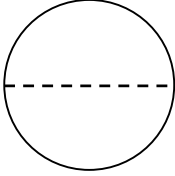
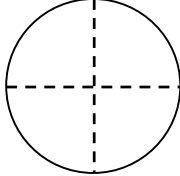
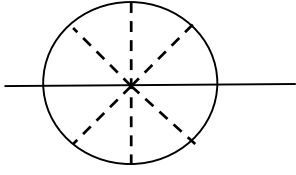
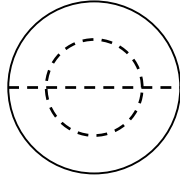
$$J_n(x) = \frac{\left(\frac{1}{2}x\right)^n}{n!} \left[1 - \frac{\left(\frac{1}{2}x\right)^2}{1(n+1)} + \frac{\left(\frac{1}{2}x\right)^4}{1.2.(n+1)(n+2)} - \dots \right]$$

From here we obtain

$$J_0(x) = 1 - \left(\frac{1}{2}x\right)^2 + \frac{\left(\frac{1}{2}x\right)^4}{2^2} + \dots$$

$$J_1(x) = \frac{1}{2}x - \frac{\left(\frac{1}{2}x\right)^3}{2} + \frac{\left(\frac{1}{2}x\right)^5}{2^2 \cdot 3} - \dots$$

Tab. 3.3.2 Natural frequencies and nodal lines of circular membrane

Ω_{0m}		Nodal lines
$\frac{2,405}{R} \sqrt{\frac{N}{q}}$	$n = 0$ $m = 1$	 Circumference of the circle
$\frac{3,832}{R} \sqrt{\frac{N}{q}}$	$n = 1$ $m = 1$	
$\frac{5,136}{R} \sqrt{\frac{N}{q}}$	$n = 2$ $m = 1$	
$\frac{6,380}{R} \sqrt{\frac{N}{q}}$	$n = 3$ $m = 1$	
$\frac{7,016}{R} \sqrt{\frac{N}{q}}$	$n = 1$ $m = 2$	

If on the beginning the membrane is in quietness the constant $D = 0$ and the solution obtains the form

$$w = \sum_{m=1}^{\infty} C_m \cos(\Omega_{0m}t) J_0(kr) \quad (3.3.64)$$

The vibration will be periodic with period $T = \frac{2\pi}{\Omega_{0m}}$ and natural circular frequency is

$$\Omega_{0m} = \frac{1}{R} \sqrt{\frac{N}{q}} J_0(k_m R) \quad (3.3.65)$$

The nodal lines are obtained from the condition $w = 0$. Some cases shows the Tab.3.3.2

3.3.5 Bending vibrations of thin plates

The objective of this section is to introduce the concept of the solution of thin plates and to analyze their free vibration behaviour.

The assumptions adopted generalize to two dimensional object are:

1. The plane is thin with thickness h and possesses a mean plane. The external layers of the plane are the planes $z = \pm \frac{1}{2}h$
2. Only the transverse displacement is considered
3. The stress σ_z in the transverse direction is zero. Indeed, it must vanish on the external layers and, since the plane is thin, it is natural to assume that it vanishes for all z
4. The cross section initially normal to the mean plane, remain plane and orthogonal to it, implying that the transverse shear strain is neglected
5. The normal stresses in the mean plane are zero

Let we consider an element of the plate dimensions dx , dy and thickness h loaded according the Fig. 3.32

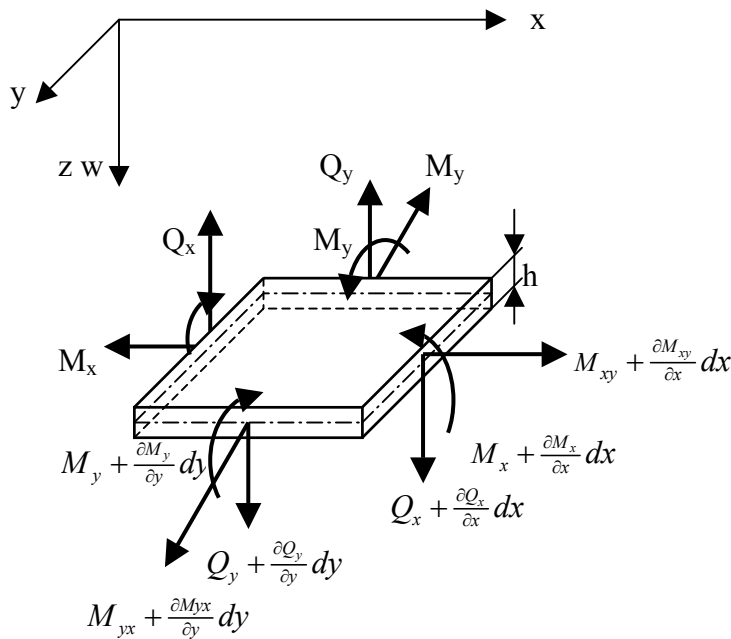


Fig. 3.32

All forces and moments are considered relative to the unit area.

The equation of transverse motion of the element describes the expression

$$\frac{\partial Q_x}{\partial x} dx dy + \frac{\partial Q_y}{\partial y} dx dy = \rho h dx dy \frac{\partial^2 w(x, y, t)}{\partial t^2}$$

from which we get

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} \quad (3.3.66)$$

Equation of motion of rotation about axis x is

$$\frac{\partial M_y}{\partial y} dx dy - \frac{\partial M_{xy}}{\partial x} dx dy - Q_y dx dy = \frac{1}{12} \rho h (dx)^3 dy \frac{\partial^2 \phi_x}{\partial t^2}$$

The order of term on the right hand side of the equation is twice smaller and we neglect it.

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \quad (3.3.67)$$

By similar way we obtain equation of rotation about the axis y

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \quad (3.3.68)$$

The internal moments and forces are described by expressions [5]:

$$\begin{aligned} M_x &= -D \left(\frac{\partial^2 w(x, y, t)}{\partial x^2} + \mu \frac{\partial^2 w(x, y, t)}{\partial y^2} \right) \\ M_y &= -D \left(\frac{\partial^2 w(x, y, t)}{\partial y^2} + \mu \frac{\partial^2 w(x, y, t)}{\partial x^2} \right) \\ M_{xy} &= -M_{yx} = D(1 - \mu) \frac{\partial^2 w(x, y, t)}{\partial x \partial y} \\ Q_x &= - \left(\frac{\partial^3 w(x, y, t)}{\partial x^3} + (2 - \mu) \frac{\partial^3 w(x, y, t)}{\partial x \partial y^2} \right) \\ Q_y &= -D \left(\frac{\partial^3 w(x, y, t)}{\partial y^3} + (2 - \mu) \frac{\partial^3 w(x, y, t)}{\partial x^2 \partial y} \right) \end{aligned} \quad (3.3.69)$$

D is the *bending stiffness* given by formula

$$D = \frac{Eh^3}{12(1 - \mu^2)} \quad (3.3.70)$$

μ is *Poisson number*. Substituting (3.3.69) into (3.3.66), (3.3.67) and (3.3.68) we obtain after

editing

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2 w(x, y, t)}{\partial x^2} + \frac{\partial^2 w(x, y, t)}{\partial y^2} \right) = -\frac{\rho h}{D} \frac{\partial^2 w(x, y, t)}{\partial t^2} \quad (3.3.71)$$

When Laplace operator will be used then

$$\nabla^2 \nabla^2 w(x, y, t) = \nabla^4 w(x, y, t) = -\frac{\rho h}{D} \frac{\partial^2 w(x, y, t)}{\partial t^2} \quad (3.3.72)$$

Next solution of these equations depend on the form of a plate and on boundary conditions.

3.3.5.1 Bending vibrations of rectangular plate

Analytical solution of plate is possible only for some types of supports. The solution is possible always, when two opposite edges are hinged. The other edges may be supported arbitrary. Let we show the solution of some cases.

Rectangular plate with all sides simply supported

The schema is shown on the Fig. 3.33. The deflection of the plate will be composed of the term depending on the position x, y and a term depending on the time:

$$w(x, y, t) = W(x, y)e^{i\Omega t}$$

The mode of vibrations must satisfy equation (3.3.72). After substituting the previous term we get

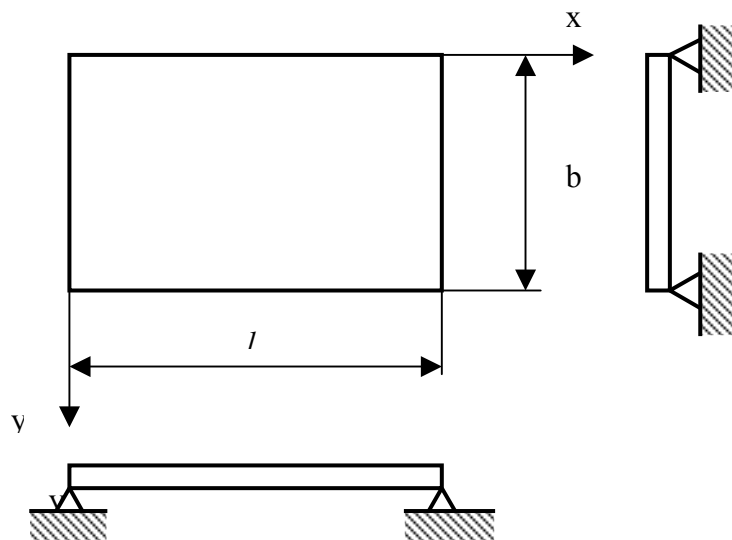


Fig. 3.33

$$\nabla^4 W(x, y) = \frac{\rho h}{D} \Omega^2 W(x, y) \quad (3.3.73)$$

The Laplace operator of 4th order is given by formula

$$\nabla^4 W(x, y) = \frac{\partial^4 W(x, y)}{\partial x^4} + 2 \frac{\partial^4 W(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y)}{\partial y^4}$$

The boundary conditions are: for $x = 0$ and $x = l$ is $W = 0$; $M_x = -D \left(\frac{\partial^2 W}{\partial x^2} + \mu \frac{\partial^2 W}{\partial y^2} \right) = 0$

$$\text{for } y = 0 \text{ and } y = b \text{ is } W = 0; M_y = -D \left(\frac{\partial^2 W}{\partial y^2} + \mu \frac{\partial^2 W}{\partial x^2} \right) = 0$$

For $W(x, y)$ we choose such function which satisfied (3.3.73) as well as the boundary conditions

$$W(x, y) = C \sin\left(\frac{m\pi}{l} x\right) \sin\left(\frac{n\pi}{b} y\right) \text{ for } m = 1, 2, \dots, \infty \text{ and } n = 1, 2, \dots, \infty \quad (3.3.74)$$

We substitute (3.3.74) in (3.3.73) and get the natural circular frequency

$$\Omega_{m,n} = \pi^2 \left(\frac{m^2}{l^2} + \frac{n^2}{b^2} \right) \sqrt{\frac{D}{\rho h}} \quad (3.3.75)$$

The deflection of plate will be given by linear combination of all solutions

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{m,n} \cos \Omega_{m,n} t + B_{m,n} \sin \Omega_{m,n} t \right) \sin \frac{m\pi}{l} x \sin \frac{n\pi}{b} y \quad (3.3.76)$$

or

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(C_{m,n} \sin(\Omega_{m,n} t + \varphi_{m,n}) \right) \sin \frac{\pi m}{l} x \sin \frac{\pi n}{b} y \quad (3.3.77)$$

The nodal lines satisfied the condition $w = 0$:

$$x = \frac{l}{m}, \frac{2l}{m}, \dots, \frac{(m-1)l}{m} \quad a \quad \frac{b}{n}, \frac{2b}{n}, \dots, \frac{(n-1)b}{n}$$

The nodal lines are in this case the same as by rectangular membrane.

Rectangular plate on two sides hinged and on other sides fixed

This case is shown in Fig 3.34. We consider the particular solution in the form

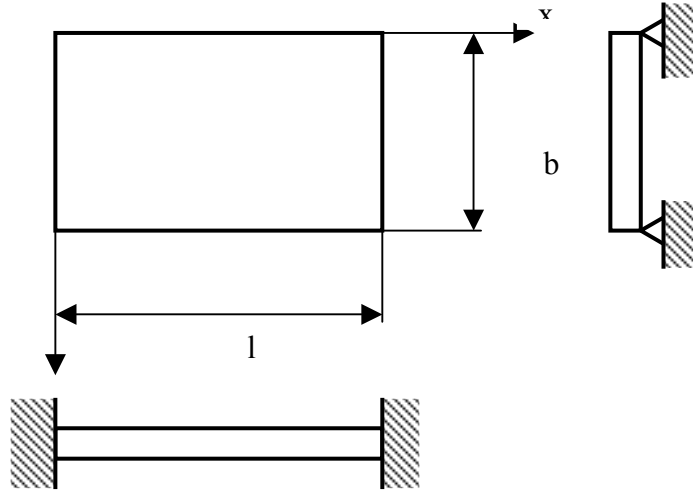


Fig. 3.34

$$W(x, y) = C_{m,n} X(x) \sin \frac{m\pi}{b} y \quad (3.3.78)$$

Substituting (3.3.78) in (3.3.66) we obtain

$$\begin{aligned} \frac{d^4 X}{dx^4} - 2 \frac{m^2 \pi^2}{b^2} \frac{d^2 X}{dx^2} + \\ + \left(\frac{m^4 \pi^4}{b^4} - \frac{\rho h}{D} \Omega_{m,n}^2 \right) X = 0 \end{aligned} \quad (3.3.79)$$

The solution is considered $X = Ae^{\lambda x}$ and after substitution in (3.3.79) we obtain characteristic equation

$$\lambda^4 - 2 \frac{m^2 \pi^2}{b^2} \lambda^2 + \left(\frac{m^4 \pi^4}{b^4} - \frac{\rho h}{D} \Omega_{m,n}^2 \right) = 0$$

From here
$$\lambda_{1,2}^2 = \frac{m^2 \pi^2}{b^2} \pm \Omega_{m,n} \sqrt{\frac{\rho h}{D}}$$

We plot on

$$\begin{aligned} r &= \sqrt{\Omega_{m,n} \sqrt{\frac{\rho h}{D}} + \frac{m^2 \pi^2}{b^2}} \\ s &= \sqrt{\Omega_{m,n} \sqrt{\frac{\rho h}{D}} - \frac{m^2 \pi^2}{b^2}} \end{aligned}$$

By using of these terms we can write

$$\lambda_1 = r; \lambda_2 = -r; \lambda_3 = is; \lambda_4 = -is$$

These values we substitute in (3.3.79). The solution gives

$$X = A_1 \cosh rx + A_2 \sinh rx + A_3 \cos sx + A_4 \sin sx \quad (3.3.80)$$

The boundary conditions for supposed plate are

$x = 0$ a $x = l$: $X = 0$; $\frac{dX}{dx} = 0$. From these boundary conditions we get:

$$A_1 + A_3 = 0 \text{ and } A_2 r + A_4 s = 0$$

Equation (3.3.80) now will be

$$X(x) = A_1 (\cosh rx - \cos sx) + A_2 \left(\sinh rx - \frac{r}{s} \sin sx \right)$$

When $x = l$ and denote $R=rl$ and $S=sl$ we obtain

$$\begin{aligned} A_1 (\cosh R - \cos S) + A_2 \left(\sinh R - \frac{R}{S} \sin S \right) &= 0 \\ A_1 \left(\sinh R + \frac{S}{R} \sin S \right) + A_2 (\cosh R - \cos S) &= 0 \end{aligned} \quad (3.3.81)$$

We obtained two homogenous equations for A_1 and A_2 . For non-trivial solution the determinant of the system must be zero. From this requirement we obtain

$$2(1 - \cosh R \cos S) + \frac{R^2 - S^2}{RS} \sinh R \sin S = 0$$

From this equation we obtain the natural circular frequencies

$$\Omega_{m,n} = \frac{r^2 + s^2}{2} \sqrt{\frac{D}{\rho h}} = \frac{R^2 + S^2}{2l^2} \sqrt{\frac{D}{\rho h}} \quad (3.3.82)$$

Rectangular plate on three sides simply supported and on the forth side clamped

This case shoves the Fig. 3.35. The deformation gives the equations (3.3.78) and (3.3.79).

The boundary conditions are

$$x = 0 \rightarrow X(x) = 0 \text{ and } \frac{d^2 X(x)}{dx^2} = 0$$

From here the integrations constants are $A_1 = A_3 = 0$ and the equation (3.3.79) obtains the form

$$X(x) = A_2 \sinh rx + A_4 \sin sx$$

At the clamped edge $x = l \rightarrow X(l) = 0; \frac{dX(l)}{dx} = 0$ and we obtain two equations

$$A_2 \sinh R + A_4 \sin S = 0$$

$$A_2 \cosh R + A_4 \frac{S}{R} \cos S = 0$$

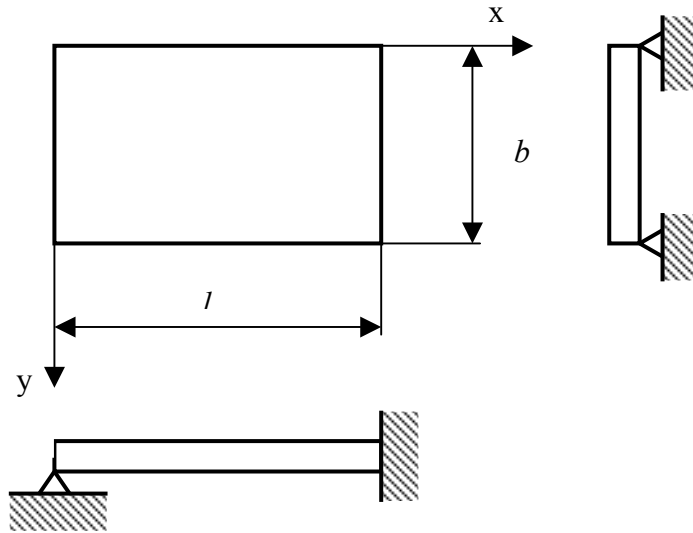


Fig. 3.35

We use the condition of non-trivial solution and from the determinate of the system we get the solution

$$S \sinh R \cos S - R \cosh R \sin S = 0$$

From this equation we determine the natural circular frequency

$$\Omega_m = \frac{m^2 \pi^2}{b^2} \sqrt{\frac{D}{\rho h}}$$

Rectangular plate on two opposite edges simply supported on two other edges free

Scheme of this case is shown on the Fig. 3.36. Determination of deflection will be calculated by using of equation (3.3.78). The boundary conditions we use at $x = 0$ and $x = l$. In these positions is valid

$$M_x(0, y) = M_x(l, y) = 0$$

$$Q_x(0, y) = Q_x(l, y) = 0$$

From (3.3.69) we obtain

$$-DC_{m,n} \left(\frac{d^2 X}{dx^2} - (2 - \mu) X \frac{m^2 \pi^2}{b^2} \right) \sin \frac{m\pi}{b} y = 0$$

$$-DC_{m,n} \left(\frac{d^3 X}{dx^3} - (2 - \mu) \frac{dX}{dx} \frac{m^2 \pi^2}{b^2} \right) \sin \frac{m\pi}{b} y = 0$$

The expressions in brackets must be zero, because the conditions are valid for any y :

$$\frac{d^2 X}{dx^2} - \mu \frac{m^2 \pi^2}{b^2} X = 0$$

$$\frac{d^3 X}{dx^3} - (2 - \mu) \frac{m^2 \pi^2}{b^2} \frac{dX}{dx} = 0$$

By using (3.3.80) it is possible to express

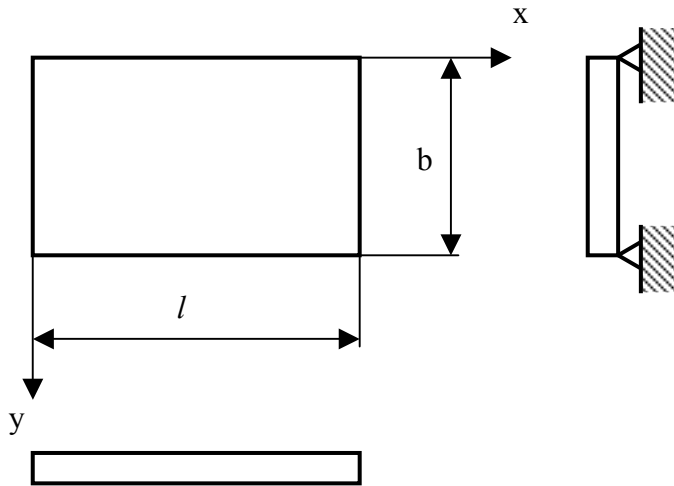


Fig. 3.36

$$A_3 = \frac{r^2 - \mu \frac{m^2 \pi^2}{b^2}}{s^2 + \mu \frac{m^2 \pi^2}{b^2}} A_1 = \alpha A_1$$

$$A_4 = \frac{r}{s} \frac{r^2 + (2 - \mu) \frac{m^2 \pi^2}{b^2}}{s^2 - (2 - \mu) \frac{m^2 \pi^2}{b^2}} A_2 = \beta A_2$$

When we use these expressions the deflection of the plate is given

$$X(x) = A_1 (\cosh rx + \alpha \cos sx) + A_2 (\sinh rx + \beta \sin sx)$$

Now we use the condition at $x = l$. We use again $R = rl$ and $S = sl$

$$\begin{aligned}
& A_1 \left[\left(r^2 - \mu \frac{m^2 \pi^2}{b^2} \right) \cosh R - \alpha \left(s^2 + \mu \frac{m^2 \pi^2}{b^2} \right) \cos S \right] + \\
& + A_2 \left[\left(r^2 - \mu \frac{m^2 \pi^2}{b^2} \right) \sinh R - \beta \left(s^2 + \mu \frac{m^2 \pi^2}{b^2} \right) \sin S \right] = 0 \\
& A_1 \left\{ r \left[r^2 - (2 - \mu) \frac{m^2 \pi^2}{b^2} \right] \sinh R + s \alpha \left[s^2 + (2 - \mu) \frac{m^2 \pi^2}{b^2} \right] \sin S \right\} + \\
& + A_2 \left\{ r \left[r^2 - (2 - \mu) \frac{m^2 \pi^2}{b^2} \right] \cosh R - s \beta \left[s^2 + (2 - \mu) \frac{m^2 \pi^2}{b^2} \right] \cos S \right\} = 0
\end{aligned}$$

From the frequency determinant we determine r and s and after that the natural circular frequencies. (The solution will be provided by some mathematical sw).

3.3.5.2 Vibrations of circular plates

We suppose circular plate with diameter R and thickness h . It is useful introduce polar coordinates. The equation (3.3.72) obtains the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} \right)^2 w(r, \vartheta) = \frac{\rho h \Omega^2}{D} w(r, \vartheta) \quad (3.3.83)$$

We make the second root of (3.3.83)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} \right) w(r, \vartheta) = \pm \Omega \sqrt{\frac{\rho h}{D}} w(r, \vartheta) \quad (3.3.84)$$

Equation (3.3.83) is fulfilled if is fulfilled one of (3.3.84). Equation of natural mode of vibrations is possible to express by

$$w_{m,n}(r, \vartheta) = \left[A_{m,n} J_m \left(\lambda_{m,n} \frac{r}{R} \right) + B_{m,n} J_m \left(i \lambda_{m,n} \frac{r}{R} \right) \right] \sin(m\vartheta + \varphi_{m,n}) \quad (3.3.85)$$

Here we introduced

$$\lambda = R \sqrt[4]{\frac{\rho h \Omega^2}{D}} \quad (3.3.86)$$

We determine natural circular frequency Ω_{mn} and λ_{mn} .

The plate clamped at circumference has the boundary conditions

$$r = R; w(R, \vartheta) = 0; \frac{\partial w(R, \vartheta)}{\partial r} = 0 \text{ and from (3.3.85) we get}$$

$$\begin{aligned}
A_{m,n}J_m(\lambda_{m,n}) + B_{m,n}J_m(i\lambda_{m,n}) &= 0 \\
A_{m,n}\frac{\partial J_m(\lambda_{m,n})}{\partial r} + B_{m,n}\frac{\partial J_m(i\lambda_{m,n})}{\partial r} &= 0
\end{aligned} \tag{3.3.87}$$

(3.3.87) enables determine amplitude ratio $\frac{A_{m,n}}{B_{m,n}}$. In case when $m = n = 0$ $\lambda_{mn} = 3,2$ and from

(3.3.86) it is possible determine natural circular frequency

$$\Omega = \frac{\lambda^2}{R^2} \sqrt{\frac{D}{\rho h}} = \frac{3,2^2}{R^2} \sqrt{\frac{D}{\rho h}}$$

3.4 Approximation of continuous system

Among the problems of elastodynamics governed by a system of partial differential equations, some of which have been considered in the previous chapters, very few have a closed-form solution which simultaneously verifies the differential equations and the boundary conditions

3.4.1 Rayleigh method

We know that Rayleigh quotient is given as the ratio of potential and unit kinetic energy

$$\lambda = \Omega^2 = \frac{E_{P \max}}{E_{K \max}^*} \tag{3.4.1}$$

By this method we can take in the solution the influence of mass points placed on linear continues and the influence of elastic supports. We bring in following part the expressions of potential and unit kinetic energy of continues defined in previous chapter.

Longitudinal vibrating bar:

$$E_p = \frac{1}{2} E \int_0^l A(x) U'^2(x) dx \tag{3.4.2}$$

$$E_K^* = \frac{1}{2} \int_0^l A(x) \rho U^2(x) dx + \frac{1}{2} \sum_{j=1}^n m_j U^2(x_j) \tag{3.4.3}$$

Torsionale vibrating shaft:

$$E_p = \frac{1}{2} G \int_0^l J_p(x) \Phi'^2(x) dx \tag{3.4.4}$$

$$E_K^* = \frac{1}{2} \rho \int_0^l J_p(x) \Phi^2(x) dx + \frac{1}{2} \sum_{j=1}^n I_j \Phi^2(x_j) \tag{3.4.5}$$

Bending vibrating beam:

$$E_p = \frac{1}{2} E \int_0^l J(x) W''^2(x) dx + \frac{1}{2} \sum_{j=1}^r k_j W^2(x_j) \tag{3.4.6}$$

$$E_K^* = \frac{1}{2} \rho \int_0^l A(x) W^2(x) dx + \frac{1}{2} \sum_{j=1}^n m_j W^2(x_j) \tag{3.4.7}$$

Vibrating rectangular membrane:

$$E_p = \frac{1}{2} N \iint_A \left\{ \left[\frac{\partial W(x, y)}{\partial x} \right]^2 + \left[\frac{\partial W(x, y)}{\partial y} \right]^2 \right\} dx dy \quad (3.4.8)$$

$$E_K^* = \frac{1}{2} q \iint_A W^2(x, y) dx dy + \sum_j^n m_j W^2(x_j, y_j) \quad (3.4.9)$$

Vibrating circular membrane

$$E_p = \frac{1}{2} N \iint_A \left\{ \left[\frac{\partial W(r, \vartheta)}{\partial r} \right]^2 + \frac{1}{r^2} \left[\frac{\partial W(r, \vartheta)}{\partial \vartheta} \right]^2 \right\} r d\vartheta dr \quad (3.4.10)$$

$$E_K^* = \frac{1}{2} q \iint_A W^2(r, \vartheta) r d\vartheta dr + \sum_{j=1}^n m_j W^2(r_j, \vartheta_j) \quad (3.4.11)$$

Vibrating Rectangular plate

$$E_p = \frac{1}{2} D \iint_A \left\{ \left[\frac{\partial^2 W(x, y)}{\partial x^2} + \frac{\partial^2 W(x, y)}{\partial y^2} \right]^2 - 2(1-\mu) \left[\frac{\partial^2 W(x, y)}{\partial x^2} \frac{\partial^2 W(x, y)}{\partial y^2} - \left(\frac{\partial^2 W(x, y)}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (3.4.12)$$

$$E_K^* = \frac{1}{2} \rho h \iint_A W^2(x, y) dx dy + \sum_{j=1}^n m_j W^2(x_j, y_j) \quad (3.4.13)$$

Vibrating circular plate

$$E_p = \frac{1}{2} D \int_{r_0}^{r_1} \left\{ \left[\frac{d^2 W(r)}{dr^2} \right]^2 + \frac{1}{r^2} \left[\frac{dW(r)}{dr} \right]^2 + 2\mu \frac{1}{r} \frac{dW(r)}{dr} \frac{d^2 W(r)}{dr^2} \right\} r dr \quad (3.4.14)$$

$$E_K^* = \frac{1}{2} \rho h \int_{r_0}^{r_1} W^2(r) r dr \quad (3.4.15)$$

These expressions are valid for axial symmetric circular plate.

Rayleigh quotient gives exact values when exact modes are used. We put in the expressions of potential and kinetic energy supposed mode, which must satisfy the boundary conditions of supports. Very good results are obtained when the static deformations are used. From all introduced expressions it is seen that gives the possibility to solve the problems with variable cross sections.

Let we show this method on the cantilever beam with constant high and variable width from b_0 at the clamped side to zero at the free end (Fig. 3.37). The cross section of the beam is given

$$A(x) = b_0 h \left(1 - \frac{x}{l} \right)$$

The quadratic moment of the cross section is

$$J(x) = \frac{1}{12} b_0 h^3 \left(1 - \frac{x}{l} \right)$$

The deflection curve we suppose parabolic

$$W(x) = ax^2$$

Potential energy will be calculated from (3.4.6):

$$E_P = \frac{1}{2} E \frac{1}{12} b_0 h^3 4a^2 \int_0^l \left(1 - \frac{x}{l}\right) dx =$$

$$= \frac{1}{12} E b_0 h^3 a^2 l$$

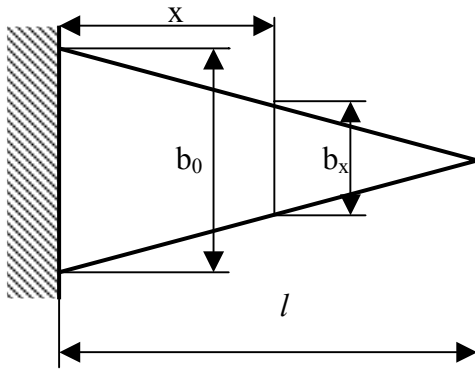


Fig. 3.37

Unit kinetic energy gives (3.4.7)

$$E_K^* = \frac{1}{2} \rho b_0 h 4a^2 \int_0^l \left(1 - \frac{x}{l}\right) x^4 dx = \frac{1}{60} \rho b_0 h a^2 l^5$$

After substituting the necessary values in (3.4.1) we obtain the natural frequency

$$\Omega^2 = \frac{E_P}{E_K^*} = \frac{\frac{1}{12} E b_0 h^3 a^2 l}{\frac{1}{60} \rho b_0 h a^2 l^5} = 5 \frac{E h^2}{\rho l^4}$$

$$\Rightarrow \Omega = 2,236 \frac{h}{l^2} \sqrt{\frac{E}{\rho}}$$

As the second case we consider the rectangular plate clamped at all sides (Fig. 3.38).

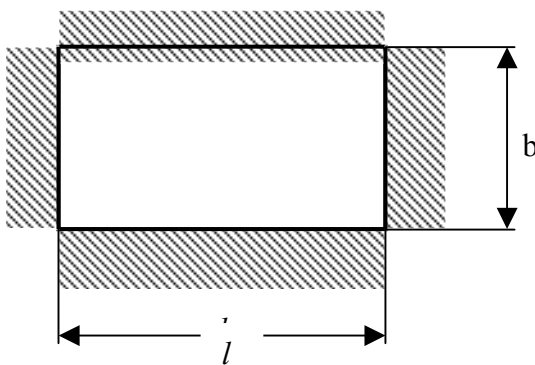


Fig. 3.38

At first we must choose such function for deflection of plate, which satisfies the boundary conditions. At clamped edges must be deflection and slope equal to zero:

$$X(x) = x^2 + a_1 x^3 + a_2 x^4$$

We determine the constants a_1 and a_2 so that the boundary condition is satisfied also for $x = l$:

$$X(l) = l^2 + a_1 l^3 + a_2 l^4 = 0$$

$$\frac{dX(l)}{dx} = 3a_1 l^2 + 4a_2 l^3 = 0$$

From here we determine the constants

$$a_1 = -\frac{2}{l} \text{ and } a_2 = \frac{1}{l^2} \text{ Using these constants}$$

the course of deflection will be given by equations

$$X(x) = x^2 - \frac{2}{l} x^3 + \frac{1}{l^2} x^4 \quad Y(y) = y^2 - \frac{2}{b} y^3 + \frac{1}{b^2} y^4$$

The general deflection of the plate is given by following equation

$$W(x, y) = \left(x^2 - \frac{2}{l} x^3 + \frac{1}{l^2} x^4 \right) \left(y^2 - \frac{2}{b} y^3 + \frac{1}{b^2} y^4 \right)$$

The shape of the deflection of the plate according this equation shows Fig. 3.39

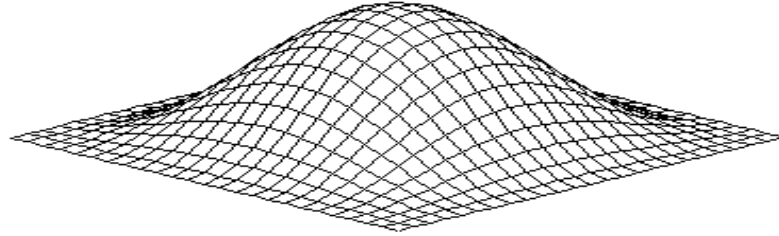


Fig. 3.39

Potential energy is according the equation (3.4.12)

$$E_p = \frac{1}{2} D \int_0^l \int_0^b \left[\left(2 - \frac{12}{l} x + \frac{12}{l^2} x^2 \right) \left(y^2 - \frac{2}{b} y^3 + \frac{1}{b^2} y^4 \right) + \left(x^2 - \frac{2}{l} x^3 + \frac{1}{l^2} x^4 \right) \left(2 - \frac{12}{l} y + \frac{12}{l^2} y^2 \right) \right]^2 dx dy =$$

$$= \frac{D}{11025} bl (7b^4 + 4l^2 b^2 + 7l^4)$$

The unit kinetic energy with respect (3.4.13) is

$$E_k = \frac{1}{2} \rho h \int_0^l \int_0^b \left(x^2 - \frac{2}{l} x^3 + \frac{1}{l^2} x^4 \right)^2 \left(y^2 - \frac{2}{b} y^3 + \frac{1}{b^2} y^4 \right)^2 dx dy = \frac{\rho h b^5 l^5}{793800}$$

After substitution in (3.3.88) we obtain

$$\Omega^2 = 72 \frac{D(7b^4 + 4l^2 b^2 + 7l^4)}{b^4 l^4 \rho h} \Rightarrow \Omega = \frac{6}{l^2 b^2} \sqrt{2(7b^4 + 4b^2 l^2 + 7l^2)} \sqrt{\frac{D}{\rho h}}$$

3.4.2 The Ritz method

Ritz method is based on the fact that Rayleigh quotient is in the interval of exact natural frequencies. Therefore the natural mode minimizes the Rayleigh quotient. Therefore one approximates natural mode by linear combination of independent functions which satisfy boundary conditions. For two dimensional continuum it will be

$$W(x, y) = a_1 f_1(x, y) + a_2 f_2(x, y) + \dots + a_n f_n(x, y) \quad (3.4.16)$$

That's mean the Rayleigh quotient will be a function of independent parameters

$$\lambda = \frac{E_p}{E_k^*} = \lambda(a_1, a_2, \dots, a_n)$$

The minima of these values are given by n conditions

$$\frac{\partial \lambda}{\partial a_i} = \frac{\partial}{\partial a_i} \left(\frac{E_p}{E_k^*} \right) = \frac{\partial}{\partial a_i} (E_p - \Omega^2 E_k^*) \quad \text{for } i = 1, 2, \dots, n \quad (3.4.17)$$

By this way n homogenous equation are obtained. For non-trivial solution the determinant of the system must be zero. Because this determinant is of n stage n natural frequencies are obtained $\Omega_1 \leq \Omega_2 \leq \dots \leq \Omega_n$. Like by all analytical methods the Ritz method is possible use only by conservative systems. The advantage is in the fact that we didn't need construct the equation of motion. Only the geometrical boundary conditions are sufficient, We show the application on the determination of first two natural frequencies of the prismatic beam built in both ends.

We propose the function, which satisfies geometrical boundary condition in the form

$$W(x) = a_1 x^2 (l-x)^2 + a_2 x^3 (l-x)^3$$

This function satisfies the conditions $W(0) = W(l) = W'(0) = W'(l) = 0$. We determine the potential energy from (3.3.93):

$$E_p = \frac{1}{2} EJ \int_0^l \left[2a_1 (l-x)^2 - 8a_1 x (l-x) + 2a_1 x^2 + 6a_2 x (l-x)^3 - 18a_2 x^2 (l-x)^2 + 6a_2 x^3 (l-x) \right]^2 dx = \\ = \frac{1}{2} EJ \left(\frac{4}{5} a_1^2 l^5 + \frac{12}{35} a_1 a_2 l^7 + \frac{2}{35} a_2^2 l^9 \right)$$

By similar way we determine from (3.3.94) kinetic energy

$$E_k^* = \frac{1}{2} \rho A \int_0^l \left[a_1 x^2 (l-x)^2 + a_2 x^3 (l-x)^3 \right]^2 dx = \frac{1}{2} \rho A \left(\frac{1}{690} a_1^2 l^9 + \frac{1}{1386} a_1 a_2 l^{11} + \frac{1}{12012} a_2^2 l^{13} \right)$$

Substituting in (3.3.104) we get after derivation two equations

$$l^5 \left[\left(\frac{4}{5} EJ - \frac{1}{630} A \rho l^4 \Omega^2 \right) a_1 + \left(\frac{6}{35} EJ l^2 - \frac{1}{2772} A \rho l^6 \Omega^2 \right) a_2 \right] = 0 \\ l^7 \left[\left(\frac{6}{35} EJ - \frac{1}{2772} A \rho l^4 \Omega^2 \right) a_1 + \left(\frac{2}{35} EJ l^2 - \frac{1}{12012} A \rho l^6 \Omega^2 \right) a_2 \right] = 0$$

We expand the frequency determinant

$$A^2 \rho^2 l^8 \Omega^4 - \frac{83952}{5} EJA \rho l^4 \Omega^2 + 8154432 E^2 J^2 = 0$$

And from here we get the natural frequencies

$$\Omega_1 = \frac{22,374}{l^2} \sqrt{\frac{EJ}{\rho A}} \quad \text{and} \quad \Omega_2 = \frac{127,632}{l^2} \sqrt{\frac{EJ}{\rho A}}$$

The exact solution gives

$$\Omega_1 = \frac{22,20}{l^2} \sqrt{\frac{EJ}{\rho A}} \quad \text{and} \quad \Omega_3 = \frac{120,90}{l^2} \sqrt{\frac{EJ}{\rho A}}$$

We see, when calculated by Ritz method 1st and 3rd natural circular frequency with mistakes 0,78% and 5,57%. Much more greater is reached by determination of shear forces and

bending moments while the base function are only geometric and force boundary conditions are not used. In these cases the mistakes are more than 100%.

3.3.6.3 The finite element method

A complete volume should be dedicated to a systematic statement of the *finite element method*. It is thus not our objective to make complete presentation of it here (it will be done in *Computational mechanics II*). Our purpose is simply to show its potential and mode of application in the context of dynamics of continuous systems. Therefore this discussion will be limited to the case of the bar in extension and the beam in bending.

The finite element method may be regarded as a particular application procedure of the Ritz method. It consists in subdividing the deformable body or the structure into a finite number of elements of simply geometry well identified structural behaviour (bar, beam, membrane, plate, shell, 3-D solid, etc.).

The interpolation functions are chosen in order to fulfil the following requirements:

1. Interpolation is performed in terms of *piecewise continuous* functions. Inside each element, the displacement field is represented by a superposition of small number of functions, which are chosen to be simple but representative of the element's structural behaviour in the global structure. They are generally of polynomial type.
2. These functions are also chosen in such a way that their intensity parameters, which are the generalized coordinates of the Ritz method, are *local values* of the displacement field in the structure.

If both conditions are strictly satisfied, the approximation obtained is kinematically admissible in the sense of the Ritz method. Indeed, the displacement field is then integrable over each element domain and imposing equal values of the generalized coordinates at element interface allows us to keep the continuity of the displacement field at the global level.

3.3.6.3.1 *The bar in extension*

a. *Generation of a bar element*

Let us consider the case of a bar in extension possibly subjected to distributed load $p(t)$. The bar is divided into N elements of length l as sketched in Fig. 3.40. The displacement field in the element is linearly interpolated by the formula

$$u(x,t) = U_1(x)T_1(t) + U_2(x)T_2(t) \quad (3.4.18)$$

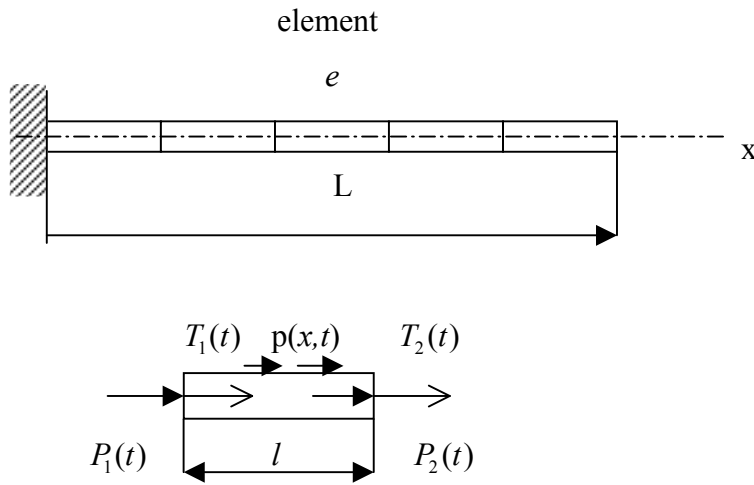


Fig. 3. 40

where are

$T_1(t), T_2(t) \dots \dots \dots$ the *connector* degrees of freedom are the axial displacements at both ends, also called *nodes*;

$U_1(x), U_2(x) \dots \dots$ are the *shape functions* of the element, chosen in such a way that

$$u(0,t) = T_1(t) \quad u(l,t) = T_2(t)$$

If no internal parameter is introduced, they result from a linear interpolation

$$U_1(x) = 1 - \frac{x}{l} \quad U_2(x) = \frac{x}{l} \quad (3.4.19)$$

Equation (3.3.105) may be put in matrix form

$$u(x,t) = \mathbf{N}_e(x)\mathbf{q}(t) \quad x \in (\text{element } e) \quad (3.4.20)$$

where

$\mathbf{N}_e(x) = [U_1(x) \ U_2(x)] \dots \dots \dots$ is the *shape function matrix* of element e

$\mathbf{q}_e^T = [T_1(t) \ T_2(t)] \dots \dots \dots$ is the set of degrees of freedom of element e

We may then determine successively

- the element kinetic energy and strain energy as quadratic forms of the mass and stiffness elementary matrices

$$E_{k,e} = \frac{1}{2} \dot{\mathbf{q}}_e^T \mathbf{M}_e \dot{\mathbf{q}}_e \quad \text{and} \quad E_{p,e} = \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e \mathbf{q}_e \quad (3.4.21)$$

Here is

$$\mathbf{M}_e = \int_0^l m \mathbf{N}_e^T \mathbf{N}_e dx \quad \mathbf{K}_e = \int_0^l EA \frac{d\mathbf{N}_e^T}{dx} \frac{d\mathbf{N}_e}{dx} dx \quad (3.4.22)$$

- the virtual work of external forces in the form

$$\delta W_e = -\delta \mathbf{q}_e^T \mathbf{g}_e(t) \quad (3.4.23)$$

$\mathbf{g}_e(t)$ is the *generalized load* conjugated to displacements $\mathbf{q}_e(t)$

$$\mathbf{g}_e(t) = \int_0^l \mathbf{N}_e^T p(x,t) dx + \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \quad (3.4.24)$$

The first term results from the discretization of the load per unit length and the second one contains the end loads of the element. The latter are themselves made of two contributions: The reaction forces with adjacent elements and the eventual external loads.

For the bar element of uniform characteristics modeled using linear interpolation functions, we obtain elementary stiffness and mass matrices

$$\mathbf{K}_e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \mathbf{M}_e = \frac{ml}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (3.4.25)$$

and the discretized force vector representing a uniform load p per unit length over the element

$$\mathbf{g}_e^{(1)} = \frac{pl}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (3.4.26)$$

Summing the elements of the mass matrix restores the total mass of the element:

$$\sum_{k,s} \mathbf{M}_{k,s} = ml$$

Hamilton's principle may be expressed in the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} \dot{\mathbf{q}}_e^T \mathbf{M}_e \dot{\mathbf{q}}_e - \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e \mathbf{q}_e \right) dt + \delta \mathbf{q}_e^T \mathbf{g}_e = 0$$

b. Assembly process

In order to express dynamic equilibrium for the global system with N elements having $N+1$ nodal displacement we suppose

$$\mathbf{q}^T = [u_0 \ u_1 \ u_2 \ \dots \ u_N] \quad (3.4.27)$$

$$\mathbf{q}_e = \mathbf{L}_e \mathbf{q} \quad (3.4.28)$$

where the *localization operator* \mathbf{L}_e is a Boolean matrix in our case with dimension $2*(N+1)$ and containing only 1 and 0 terms. For instance, for elements 1 and 2 :

$$\mathbf{L}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \mathbf{L}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}$$

By summing all the elements of the system, the structural variational equation becomes

$$\delta \int_{t_1}^{t_2} \sum_{e=1}^N \left(\frac{1}{2} \dot{\mathbf{q}}_e^T \mathbf{M} \dot{\mathbf{q}}_e - \frac{1}{2} \mathbf{q}_e^T \mathbf{K} \mathbf{q}_e \right) dt + \int_{t_1}^{t_2} \sum_{e=1}^N \delta \mathbf{q}_e^T \mathbf{g}_e dt = 0 \quad (3.4.29)$$

This equation can be expressed in terms of structural displacements through substitution of ((3.4.28) into (3.4.29)

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2} \dot{\mathbf{q}}^T \left(\sum_{e=1}^N \mathbf{L}_e^T \mathbf{M}_e \mathbf{L}_e \right) \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T \left(\sum_{e=1}^N \mathbf{L}_e^T \mathbf{K}_e \mathbf{L}_e \right) \mathbf{q} \right\} dt + \int_{t_1}^{t_2} \delta \mathbf{q}^T \left(\sum_{e=1}^N \mathbf{L}_e^T \mathbf{g}_e \right) dt = 0 \quad (3.4.30)$$

We then define

- The mass matrix of the assembled system, or structural mass matrix

$$\mathbf{M} = \sum_{e=1}^N \mathbf{L}_e^T \mathbf{M}_e \mathbf{L}_e \quad (3.4.31)$$

- the structural stiffness matrix

$$\mathbf{K} = \sum_{e=1}^N \mathbf{L}_e^T \mathbf{K}_e \mathbf{L}_e \quad (3.4.32)$$

- the structural load vector

$$\mathbf{g} = \sum_{e=1}^N \mathbf{L}_e^T \mathbf{g}_e \quad (3.4.33)$$

It is important that expressions (3.4.31) – (3.4.33) correspond to a formal representation of the assembly operation. In practice, structural assembly may be performed much more simply, by addressing correctly the matrices \mathbf{K}_e and \mathbf{M}_e in the structural matrices \mathbf{K} and \mathbf{M} (Fig.3.41).

It is observed that:

- The shaded zone corresponds to the clamped end of the bar and must then be wiped out.
- The diagonal mass and stiffness terms are two by two on the diagonal of the structural matrix.
- Owing to the system topology and the sequential numbering of the degrees of freedom, both \mathbf{K} and \mathbf{M} have tridiagonal form

Taking into account equations (3.4.33) the dynamic equilibrium equation at node j , ($0 < j < N$) is found to have a general form

$$\frac{ml}{6}(\ddot{u}_{j-1} + 4\ddot{u}_j + \ddot{u}_{j+1}) + \frac{EA}{l}(u_{j-1} + 4u_j + u_{j+1}) = g_j(t) \quad (3.4.35)$$

3.3.6.3.2 Bending vibration of beams

Let us consider the case of a beam represented in Fig. 3.42, excited by distributed load $p(x, t)$.

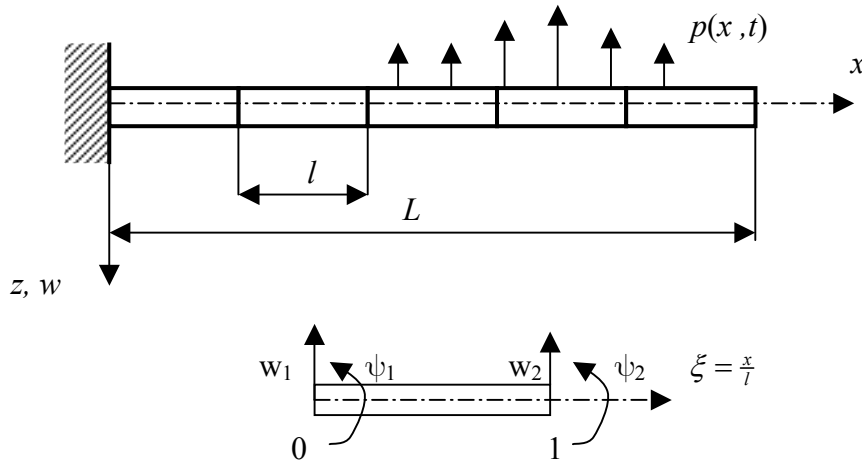


Fig. 3.42

The potential strain energy will be obtained by integration over the beam element

$$E_{p,e} = \int_0^l EJ \left(\frac{\partial^2 w(x,t)}{\partial x^2} \right)^2 dx$$

the function $w(x,t)$ and its first derivative must be continuous. Therefore, to obtain a finite element approximation of pure displacement type in the Ritz sense, the interpolation of the bending deflection must be at least cubic in order to maintain continuity of the deflection w and slope $\psi = \frac{\partial w}{\partial x}$ through nodal identification. The connectors of the element are the

deflection and slope values at both ends. In terms of the non dimensional variable $\xi = \frac{x}{l}$ over

the element domain, the cubic approximation to the deflection may be written in the form

$$w(\xi) = w_1 N_1(\xi) + \psi_1 N_2(\xi) + w_2 N_3(\xi) + \psi_2 N_4(\xi) = \mathbf{N}_e(\xi) \mathbf{q}_e(t) \quad (3.4.36)$$

$N_i(\xi)$ are the *shape functions* and they are the third-order Hermitian polynomials, matching the conditions

$$\begin{aligned} N_1(0) = 1 \quad N_1'(0) = 0 \quad N_1(1) = N_1'(1) = 0 \quad N_3(\xi) = N_1'(1-\xi) \\ N_2(0) = 0 \quad N_2'(0) = l \quad N_2(1) = N_2'(1) = 0 \quad N_4(\xi) = -N_2'(1-\xi) \end{aligned} \quad (3.4.37)$$

Here was used $N_i' = \frac{dN_i}{d\xi}$. In this manner we obtain the matrix of shape functions

$$\mathbf{N}_e^T(\xi) = \begin{bmatrix} 1-3\xi^2+2\xi^3 \\ l\xi(1-\xi)^2 \\ \xi^2(3-2\xi) \\ l\xi^2(\xi-1) \end{bmatrix} \quad (3.4.38)$$

associated with the element degrees of freedom

$$\mathbf{q}_e^T = [w_1 \quad \psi_1 \quad w_2 \quad \psi_2] \quad (3.4.39)$$

One computes successively

- The kinetic energy of the element

$$E_{k,e} = \frac{1}{2} \dot{\mathbf{q}}_e^T \mathbf{M}_e \dot{\mathbf{q}}_e \quad (3.4.40)$$

with the elementary mass matrix

$$\int_0^l m(\xi) \mathbf{N}_e^T(\xi) \mathbf{N}_e(\xi) l d\xi \quad (3.4.41)$$

- The potential strain energy of the element

$$E_{p,e} = \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e \mathbf{q}_e \quad (3.4.42)$$

with the elementary stiffness matrix

$$\mathbf{K}_e = \int_0^l EJ(\xi) \left(\frac{d^2 \mathbf{N}_e}{d\xi^2} \right)^T \left(\frac{d^2 \mathbf{N}_e}{d\xi^2} \right) \frac{d\xi}{l^3} \quad (3.4.43)$$

- The virtual work of external loads

$$\delta W_e = -\delta \mathbf{q}_e^T \mathbf{g}_e(t) \quad (3.4.44)$$

with the vector of external loads

$$\mathbf{g}_e(t) = \int_0^l \mathbf{N}_e^T(\xi) p(x,t) l d\xi \quad (3.4.45)$$

For an element of uniform characteristics solicited by a constant distributed load p_0 we explicitly obtain

$$\mathbf{K}_e = \frac{EJ}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad (3.4.46)$$

$$\mathbf{M}_e = \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

$$\mathbf{g}_e^T = \frac{p_0 l}{2} \begin{bmatrix} 1 & \frac{l}{6} & 1 & -\frac{l}{6} \end{bmatrix}$$

It is easily verified that the quadratic form $\mathbf{q}^T \mathbf{M} \mathbf{q}$ is equal to the translation inertia ml for $\mathbf{q}^T = [1 \ 0 \ 1 \ 0]$, and to the rotary inertia $\frac{ml^3}{12}$ about the centre of mass for $\mathbf{q}^T = [-\frac{l}{2} \ 1 \ \frac{l}{2} \ 1]$.

4. DIRECT INTEGRATION METHODS

Many numerical integration methods are used for the approximate solution of equation of motion or sets of such equations. A complete coverage of numerical integration methods is beyond the scope of this book and the student is referred to many available textbooks on the subject [1],[2],[3]. In this chapter we discuss some widely used step-by-step numerical integration schemes for linear and nonlinear dynamic analysis.

In a direct integration methods the equations are integrated successively using a step-by-step numerical integration procedure. The direct integration method implies that no transformation of the equations into a different form is carried out prior to integration. In direct integration methods, time derivatives are generally approximated using difference formulas involving one or more increments of time. There are two basic approaches used in the direct integration methods - explicit and implicit. In an explicit formulation the response quantities are expressed in terms of previously determined values of displacement, velocity, or acceleration. In an implicit formulation, the temporal difference equations are combined with the equations of motion, and displacements are calculated directly by solving these equations.

4.1 Explicit methods

4.1.1 Central difference method

We consider a displacement-time history curve shown in Fig. 4.1.

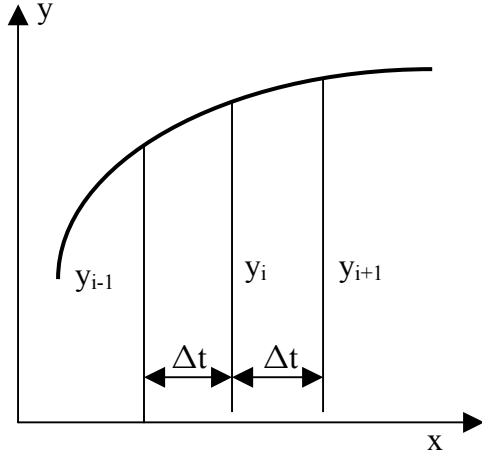


Fig. 4.1

By this method we work with finite differences instead of derivatives

$$\frac{dy}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}$$

If the curve is continuous with small change of slope, the solution is enough accurate even by greater time intervals.

The velocity in the middle of the time interval Δt is given by

$$\left(\frac{d^2 y}{dt^2}\right)_{t=t_i} = \frac{\left(\frac{dy}{dt}\right)_{t_i + \frac{\Delta t}{2}} - \left(\frac{dy}{dt}\right)_{t_i - \frac{\Delta t}{2}}}{\Delta t} = \frac{\frac{y_{i+1} - y_i}{\Delta t} - \frac{y_i - y_{i-1}}{\Delta t}}{\Delta t} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta t^2} \quad (4.1.2)$$

Substituting (4.1.1) and (4.1.2) into the equation of motion

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}_t$$

we obtain

$$\mathbf{M} \frac{\mathbf{q}_{t+\Delta t} - 2\mathbf{q}_t + \mathbf{q}_{t-\Delta t}}{\Delta t^2} + \mathbf{B} \frac{\mathbf{q}_{t+\Delta t} - \mathbf{q}_{t-\Delta t}}{2\Delta t} + \mathbf{K}\mathbf{q}_t = \mathbf{Q}_t \quad (4.1.3)$$

This equation is possible arrange to the form

$$\left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{B}\right)\mathbf{q}_{t+\Delta t} = \mathbf{Q}_t - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right)\mathbf{q}_t - \left(\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{B}\right)\mathbf{q}_{t-\Delta t} \quad (4.1.4)$$

We can denote

$$\mathbf{A} = \frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{B} \quad \bar{\mathbf{Q}} = \mathbf{Q}_t - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right)\mathbf{q}_t - \left(\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{B}\right)\mathbf{q}_{t-\Delta t} \quad (4.1.5)$$

Using these equations we can calculate the displacement in the time $t + \Delta t$

$$\mathbf{q}_{t+\Delta t} = \mathbf{A}^{-1}\bar{\mathbf{Q}}_t \quad (4.1.6)$$

And from previous equations also the velocity and acceleration by using displacements in time t and $t - \Delta t$. In next step we put $t = t + \Delta t$, $\mathbf{q}_{t-\Delta t} = \mathbf{q}_t$, $\dot{\mathbf{q}}_t = \dot{\mathbf{q}}_{t+\Delta t}$ and we determine again displacement, velocity and acceleration.

Thus, to obtain the solution at the beginning of the procedure a special starting procedure is needed, because we have not the value $q_{0-\Delta t}$. We rewrite the equations (4.1.1) and (4.1.2)

$$\begin{aligned} 2\dot{\mathbf{q}}_0\Delta t &= \mathbf{q}_{0+\Delta t} - \mathbf{q}_{0-\Delta t} \\ \ddot{\mathbf{q}}_0\Delta t^2 &= \mathbf{q}_{0+\Delta t} + 2\mathbf{q}_0 + \mathbf{q}_{0-\Delta t} \end{aligned}$$

The values \mathbf{q}_0 and $\dot{\mathbf{q}}_0$ are given by initial conditions. From both equations we exclude $\mathbf{q}_{0-\Delta t}$ and we get

$$\mathbf{q}_{0+\Delta t} = \mathbf{q}_0 + \dot{\mathbf{q}}_0\Delta t + \frac{1}{2}\ddot{\mathbf{q}}_0\Delta t^2$$

Acceleration $\ddot{\mathbf{q}}_0$ we determine from the equation of motion:

$$\ddot{\mathbf{q}}_0 = \mathbf{M}^{-1}\mathbf{Q}_0 - \mathbf{B}\dot{\mathbf{q}}_0 - \mathbf{K}\mathbf{q}_0$$

Substituting in the previous equation we get

$$\mathbf{q}_{0+\Delta t} = \mathbf{q}_0 + \dot{\mathbf{q}}_0\Delta t + \mathbf{M}^{-1}\mathbf{Q}_0 - \mathbf{B}\dot{\mathbf{q}}_0 - \mathbf{K}\mathbf{q}_0 \frac{\Delta t^2}{2} \quad (4.1.7)$$

After this starting step the solution continues according the original procedure.

The local truncation error of the difference formulas used in this method is of the order Δt^2 Time step for linear dynamic analysis is limited by the highest frequency of the system (i.e., Ω_{\max}) such that

$$\Delta t \leq \frac{0,2}{\Omega_{\max}} \quad (4.2.8)$$

When Δt does not satisfy this equation a spurious growth of the solution occurs. This is known as the numerical instability. For dynamic analysis, (4.2.8) is the necessary and sufficient condition for the stability of the central difference method.

4.1.2 Two- cycles iteration method

The incremental form of equation of motion at any time t is expressed

$$\mathbf{M}\Delta\ddot{\mathbf{q}} = \Delta\mathbf{Q}_t - \mathbf{K}\Delta\mathbf{q}_t - \mathbf{B}\Delta\dot{\mathbf{q}} \quad (4.2.9)$$

In the first iteration cycle, increments in velocities and displacements are estimated using the following formulas

For first time step

$$\Delta \dot{\mathbf{q}}_t = \Delta t \ddot{\mathbf{q}}_{t-\Delta t} \quad (4.2.10)$$

For other time step

$$\begin{aligned} \Delta \dot{\mathbf{q}}_t &= 2\Delta t \ddot{\mathbf{q}}_{t-\Delta t} - \Delta \dot{\mathbf{q}}_{t-\Delta t} \\ \dot{\mathbf{q}}_t &= \dot{\mathbf{q}}_{t-\Delta t} + \Delta \dot{\mathbf{q}}_t \\ \Delta \mathbf{q}_t &= \frac{\Delta t}{2} (\dot{\mathbf{q}}_{t-\Delta t} + \dot{\mathbf{q}}_t) \end{aligned} \quad (4.2.11)$$

Increments of acceleration are evaluated, by substituting the relations (4.2.10) and (4.2.11) in (4.2.9)

$$\begin{aligned} \Delta \ddot{\mathbf{q}}_t &= \mathbf{M}^{-1} (\Delta \mathbf{Q}_t - \mathbf{K} \Delta \mathbf{q}_t - \mathbf{B} \Delta \dot{\mathbf{q}}_t) \\ \ddot{\mathbf{q}}_t &= \ddot{\mathbf{q}}_{t-\Delta t} + \Delta \ddot{\mathbf{q}}_t \end{aligned} \quad (4.2.12)$$

In the second iteration cycle, increments in the velocities and accelerations are determined as follows:

$$\begin{aligned} \Delta \dot{\mathbf{q}}_t &= \frac{\Delta t}{2} (\ddot{\mathbf{q}}_{t-\Delta t} + \ddot{\mathbf{q}}_t) \\ \dot{\mathbf{q}}_t &= \dot{\mathbf{q}}_{t-\Delta t} + \Delta \dot{\mathbf{q}}_t \\ \Delta \mathbf{q}_t &= \frac{\Delta t}{2} (\dot{\mathbf{q}}_{t-\Delta t} + \dot{\mathbf{q}}_t) \end{aligned} \quad (4.2.13)$$

The relations from (4.2.13) are substituted in (4.2.12) to calculate the new increments in the accelerations. These are then used in (4.2.12) to evaluate accelerations at time t .

4.2.3 Runge-Kutta methods

In this method, the system equations are replaced in *state –variables* form, that is both displacements and velocities are replaced as unknowns defined by

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$$

The equation of motion is now rewritten as

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1} \mathbf{K} \mathbf{q} - \mathbf{M}^{-1} \mathbf{B} \dot{\mathbf{q}} - \mathbf{M}^{-1} \mathbf{Q}(t)$$

Using the identity

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}$$

both equations are written as

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{Q}(t) \end{bmatrix} \quad (4.2.14)$$

or

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{x} + \mathbf{Q}^*(t)$$

or more schematic

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), t) \quad (4.2.15)$$

In Runge-Kutta method, an approximation to $\mathbf{x}_{t+\Delta t}$ is obtained from \mathbf{x}_t in such a way that the power series expansion of the approximation coincides, up to terms of a certain order $(\Delta t)^N$ in the time interval Δt , with the actual Taylor series expansion of $((t + \Delta t)$ in powers of Δt . However, the method is self-starting and also has the advantage that no initial values are needed beyond the prescribed values.

For simple writing we will consider the system with one degree of freedom.

We consider that the function (4.2.15) about the point exists and is unique in the interval Δt about the point. The Taylor series expansion of the solutions yields

$$x(t + \Delta t) = x_{t+\Delta t} = x(t) + \Delta t \dot{x}(t) + \frac{(\Delta t)^2}{2!} \ddot{x}(t) + \frac{(\Delta t)^3}{3!} \dddot{x}(t) + \dots \quad (4.2.16)$$

Since we consider $\dot{x} = f(x(t), t) = f$ and further differentiation yields

$$\ddot{x}(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} = f_t + ff_x$$

Similarly

$$\dddot{x}(t) = f_{tt} + 2ff_{tx} + f^2 f_{xx} + f_x(f_t + ff_x)$$

Substituting these results in (4.2.16), we obtain

$$x(t + \Delta t) = x(t) + \Delta t f + \frac{(\Delta t)^2}{2} (f_t + ff_x) + \frac{(\Delta t)^3}{6} [f_{tt} + 2ff_{tx} + f^2 f_{xx} + f_x(f_t + ff_x)] + \dots \quad (4.2.17)$$

It has been also assumed in the following that the higher derivatives and partial derivatives exist at point required. The simplest of the Runge-Kutta method is the first order method, also known as Euler method, which retains only the first two terms of the Taylor series expansion. Hence, in the Euler method, the approximation to the solution is given by

$$x(t + \Delta t) = x(t) + \Delta t f(x(t), t) \quad (4.2.18)$$

The results are reasonably accurate only for the first few time steps with small Δt . After that the approximation usually diverges from the actual solution.

The general idea behind the higher order Runge-Kutta methods is to retain the higher order terms in (4.2.17). However, the method does not require evaluation of the derivatives of the function f . Instead, approximations are obtained at the expense of several evaluations of the function f at each time step.

The solution can also be written in the integral form

$$x(t + \Delta t) = x(t) + \int_t^{t+\Delta t} f(x(\tau), \tau) d\tau \quad (4.2.19)$$

Application of the mean value theorem of integral calculus to (4.2.19) yields

$$x(t + \Delta t) = x(t) + \Delta t f(x(t + \alpha \Delta t), t + \alpha \Delta t) \quad (4.2.20)$$

for some α such that $0 < \alpha < 1$. The problem is now to avoid the evaluation of explicit higher derivatives required in (4.2.17) and in the expansion of (4.2.20).

4.2.3.1 Second order Runge –Kutta method

Here, α is chosen so that Taylor series expansion of (4.2.20) agrees exactly with (4.2.17) up to terms of order $(\Delta t)^2$. Letting $x(t + \alpha \Delta t) = x(t) + \beta \Delta t + \dots$, the Taylor series expansion of (4.2.20) gives

$$x(t + \Delta t) = x(t) + \Delta t f + \alpha (\Delta t)^2 f_t + \beta (\Delta t)^2 f_x \quad (4.2.21)$$

Comparing (4.2.21) with (4.2.17) when only terms of order $(\Delta t)^2$ are retained, we obtain

$$\alpha = \beta = \frac{1}{2}$$

Hence, in second order Runge-Kutta method, the approximation to the solution is given by

$$x(t + \Delta t) = x(t) + \Delta t f \left(x(t) + \frac{\Delta t}{2} f(x(t), t + \frac{\Delta t}{2}) \right) \quad (4.2.22)$$

In the algorithm of numerical solution is advantageous to use the following practice

At the time t_0 is known $x(t_0) = x_0$ and we solve

$$\begin{aligned} k_1 &= \Delta t \cdot f(x_0, t_0) \\ k_2 &= \Delta t \cdot f(x_0 + k_1, t_0 + \Delta t) \\ x(t_0 + \Delta t) &= x_0 + \frac{k_1}{2} + k_2 \end{aligned} \quad (4.2.23)$$

4.2.3.2 Fourth order Runge – Kutta method

To obtain good accuracy, the commonly employed method is the fourth order Runge-Kutta method. Again, to avoid the evaluation of explicit higher order derivatives, we set

$$\begin{aligned}k_1 &= \Delta t \cdot f(x_i, t_i) \\k_2 &= \Delta t \cdot f\left(x_i + \frac{k_1}{2}, t_i + \frac{\Delta t}{2}\right) \\k_3 &= \Delta t \cdot f\left(x_i + \frac{k_2}{2}, t_i + \frac{\Delta t}{2}\right) \\k_4 &= \Delta t \cdot f(x_i + k_3, t_i + \Delta t)\end{aligned}\tag{4.2.24}$$

By using of these coefficients we get

$$x(t_i + \Delta t) = x(t_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\tag{4.2.25}$$

The first and second order Runge – Kutta methods are hardly ever employed because the results that they yield are not very accurate. Hence, if a Runge-Kutta method is chosen as the integration technique, it is usually the fourth and higher order method.

The truncation error e , for the fourth order is of the form

$$e = k(\Delta t)^5$$

where k depends on the $f(t, x)$ and its higher order partial derivatives. Since the Runge-Kutta method is an explicit method, the maximum time step is usually governed by stability considerations. The method can be considered as an inherently stable method, since the change in time step can be easily implemented at any stage of the advance calculation. The principal disadvantage consists in the fact that each forward step requires several evaluations of the functions. This increases considerable the time and cost of computation.

4.2 Implicit methods

From all implicit methods we show three of them.

4.2.1 Houbolt method

This method is based on a third order interpolation of displacements. In the Houbolt integration scheme, multistep implicit formulas for velocity and acceleration are derived in terms of displacements using backward differences. With references to Fig. 4.2 we can write

$$q_t = q_{t+\Delta t} - \Delta t \dot{q}_{t+\Delta t} + \frac{\Delta t^2}{2} \ddot{q}_{t+\Delta t} - \frac{\Delta t^3}{6} \dddot{q}_{t+\Delta t} \quad \begin{array}{l} | \text{(-8)} \\ | \text{(-27)} \end{array} \quad (4.2.1a)$$

$$q_{t-\Delta t} = q_{t+\Delta t} - 2\Delta t \dot{q}_{t+\Delta t} + \frac{(2\Delta t)^2}{2} \ddot{q}_{t+\Delta t} - \frac{(2\Delta t)^3}{6} \dddot{q}_{t+\Delta t} \quad (4.2.1b)$$

$$q_{t-2\Delta t} = q_{t+\Delta t} - 3\Delta t \dot{q}_{t+\Delta t} + \frac{(3\Delta t)^2}{2} \ddot{q}_{t+\Delta t} - \frac{(3\Delta t)^3}{6} \dddot{q}_{t+\Delta t} \quad (4.2.1c)$$

Solving equations (4.2.1 a, b, c) for $\ddot{q}_{t+\Delta t}$ and $\dot{q}_{t+\Delta t}$ in terms of $q_{t+\Delta t}$, q_t , $q_{t-\Delta t}$ we obtain the following formulas:

$$\ddot{q}_{t+\Delta t} = \frac{1}{\Delta t^2} (2q_{t+\Delta t} - 5q_t + 4q_{t-\Delta t} - q_{t-2\Delta t}) \quad (4.2.2)$$

$$\dot{q}_{t+\Delta t} = \frac{1}{6\Delta t} (11q_{t+\Delta t} - 18q_t + 9q_{t-\Delta t} - 2q_{t-2\Delta t}) \quad (4.2.3)$$

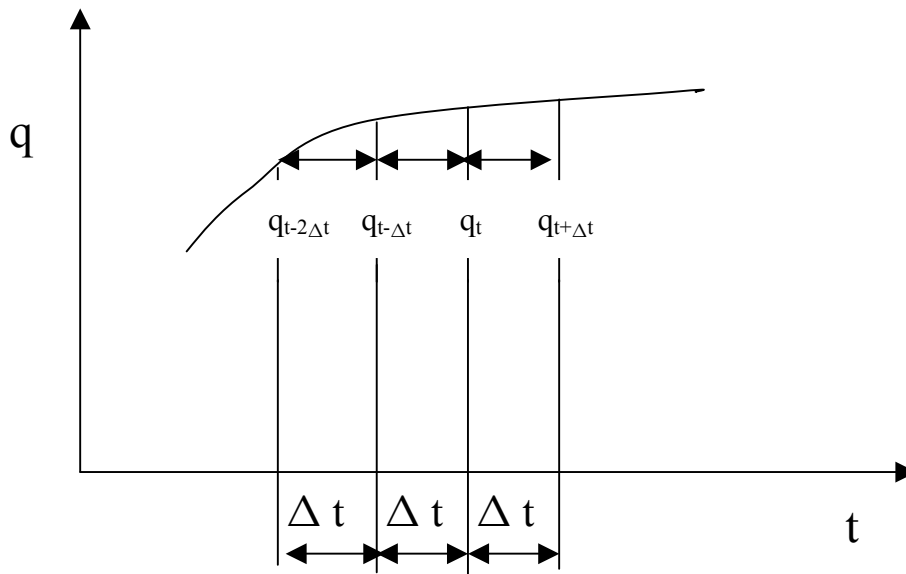


Fig. 4.2

The same form of equations is in matrix form:

$$\ddot{\mathbf{q}}_{t+\Delta t} = \frac{1}{\Delta t^2} (2\mathbf{q}_{t+\Delta t} - 5\mathbf{q}_t + 4\mathbf{q}_{t-\Delta t} - \mathbf{q}_{t-2\Delta t}) \quad (4.2.4)$$

$$\dot{\mathbf{q}}_{t+\Delta t} = \frac{1}{6\Delta t} (11\mathbf{q}_{t+\Delta t} - 18\mathbf{q}_t + 9\mathbf{q}_{t-\Delta t} - 2\mathbf{q}_{t-2\Delta t}) \quad (4.2.5)$$

Substituting (4.2.4) and (4.2.5) into the equation of motion we obtain

$$\begin{aligned} & \mathbf{M} \frac{1}{\Delta t^2} (2\mathbf{q}_{t+\Delta t} - 5\mathbf{q}_t + 4\mathbf{q}_{t-\Delta t} - \mathbf{q}_{t-2\Delta t}) + \\ & + \mathbf{B} \frac{1}{6\Delta t} (11\mathbf{q}_{t+\Delta t} - 18\mathbf{q}_t + 9\mathbf{q}_{t-\Delta t} - 2\mathbf{q}_{t-2\Delta t}) + \mathbf{K}\mathbf{q}_{t+\Delta t} = \mathbf{Q}_{t+\Delta t} \end{aligned}$$

We arrange this equation in the form:

$$\begin{aligned} & \left(\mathbf{M} \frac{2}{\Delta t^2} + \mathbf{B} \frac{11}{6\Delta t} + \mathbf{K} \right) \mathbf{q}_{t+\Delta t} - \left(\mathbf{M} \frac{5}{\Delta t^2} + \frac{3}{\Delta t} \mathbf{B} \right) \mathbf{q}_t + \\ & + \left(\mathbf{M} \frac{4}{\Delta t^2} + \mathbf{B} \frac{3}{2\Delta t} \right) \mathbf{q}_{t-\Delta t} - \left(\mathbf{M} \frac{1}{\Delta t^2} + \mathbf{B} \frac{1}{3\Delta t} \right) \mathbf{q}_{t-2\Delta t} = \mathbf{Q}_{t+\Delta t} \end{aligned} \quad (4.2.6)$$

We denote the effective mass matrix $\bar{\mathbf{M}}$ and effective force vector $\bar{\mathbf{Q}}_{t+\Delta t}$

$$\bar{\mathbf{M}} = \left(\frac{2}{\Delta t^2} \mathbf{M} + \frac{11}{6\Delta t} \mathbf{B} + \mathbf{K} \right) \quad (4.2.7)$$

$$\begin{aligned} \bar{\mathbf{Q}}_{t+\Delta t} = & \mathbf{Q}_{t+\Delta t} + \left(\frac{5}{\Delta t^2} \mathbf{M} + \frac{3}{2\Delta t} \mathbf{B} \right) \mathbf{q}_t - \\ & - \left(\frac{4}{\Delta t^2} \mathbf{M} + \frac{3}{2\Delta t} \mathbf{B} \right) \mathbf{q}_{t-\Delta t} + \left(\frac{1}{\Delta t^2} \mathbf{M} + \frac{1}{3\Delta t} \mathbf{B} \right) \mathbf{q}_{t-2\Delta t} \end{aligned} \quad (4.2.8)$$

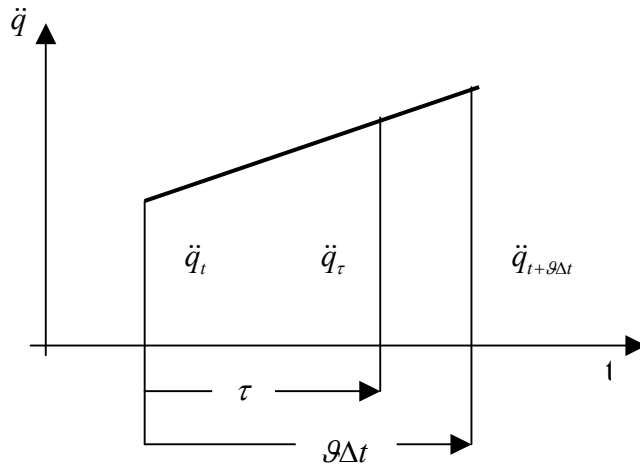
Now it is possible determine the displacement

$$\mathbf{q}_{t+\Delta t} = \bar{\mathbf{M}}^{-1} \bar{\mathbf{Q}}_{t+\Delta t} \quad (4.2.9)$$

It can be noticed that in the Houbolt method, calculation of $\mathbf{q}_{t+\Delta t}$ involves displacements at $t, t - \Delta t, t - 2\Delta t$. Therefore a special starting procedure is required to obtain solution at time Δt and $2\Delta t$. The method also requires large computer storage to store displacements for two previous time steps.

4.2.2 Wilson theta method

In the Wilson theta method, it is assumed that the acceleration varies linearly over an increment of time $\mathcal{G}\Delta t$, where $\mathcal{G} \geq 1,0$ as shown in Fig. 4.3.



The dynamic system remains constant during the interval. If τ is the time increase between t and $t + 9\Delta t$, then for the time interval t to $t + 9\Delta t$, it is assumed that

Fig. 4.3

$$\frac{\ddot{q}_{t+9\Delta t} - \ddot{q}_t}{\ddot{q}_{t+\tau} - \ddot{q}_t} = \frac{9\Delta t}{\tau} \Rightarrow \ddot{q}_{t+\tau} = \ddot{q}_t + \frac{\tau}{9\Delta t} (\ddot{q}_{t+9\Delta t} - \ddot{q}_t) \quad (4.2.10)$$

Integrating (4.2.10) we obtain

$$\int_{\dot{q}_t}^{\dot{q}_{t+\tau}} \ddot{q}_{t+\tau} d\dot{q} = \int_0^{\tau} \ddot{q}_t d\tau + \frac{1}{9\Delta t} (\ddot{q}_{t+9\Delta t} - \ddot{q}_t) \int_0^{\tau} \tau d\tau$$

$$\dot{q}_{t+\tau} = \dot{q}_t + \ddot{q}_t \tau + \frac{\tau^2}{2\Delta t} (\ddot{q}_{t+9\Delta t} - \ddot{q}_t) \quad (4.2.11)$$

After the next integration

$$q_{t+\tau} = q_t + \dot{q}_t \tau + \frac{\ddot{q}_t}{2} \tau^2 + \frac{\tau^3}{6\Delta t} (\ddot{q}_{t+9\Delta t} - \ddot{q}_t) \quad (4.2.12)$$

Substituting $\tau = 9\Delta t$ into (4.2.11) and (4.2.12)

we obtain following expression at time $t + 9\Delta t$:

$$\dot{q}_{t+9\Delta t} = \dot{q}_t + \ddot{q}_t 9\Delta t + \frac{9\Delta t}{2} (\ddot{q}_{t+9\Delta t} - \ddot{q}_t) \quad (4.2.13)$$

$$q_{t+9\Delta t} = q_t + \dot{q}_t 9\Delta t + \frac{\ddot{q}_t}{2} (9\Delta t)^2 + \frac{(9\Delta t)^2}{6} (\ddot{q}_{t+9\Delta t} - \ddot{q}_t) \quad (4.2.14)$$

From these equations we obtain

$$\dot{q}_{t+9\Delta t} = \frac{3}{9\Delta t} (q_{t+9\Delta t} - q_t) - 2\dot{q}_t - \frac{\ddot{q}_t 9\Delta t}{2} \quad (4.2.15)$$

$$\ddot{q}_{t+\vartheta\Delta t} = \frac{6}{(\vartheta\Delta t)^2}(q_{t+\vartheta\Delta t} - q_t) - \frac{6\dot{q}_t}{\vartheta\Delta t} - 2\ddot{q}_t \quad (4.2.16)$$

Equations (4.2.15) and (4.2.16) are solved for $\ddot{q}_{t+\vartheta\Delta t}$ and $\dot{q}_{t+\vartheta\Delta t}$ in terms of $q_{t+\vartheta\Delta t}$ as

$$\dot{q}_{t+\vartheta\Delta t} = \frac{3}{\vartheta\Delta t}(q_{t+\vartheta\Delta t} - q_t) - 2\dot{q}_t - \frac{\ddot{q}_t\vartheta\Delta t}{2} \quad (4.2.17)$$

$$\ddot{q}_{t+\vartheta\Delta t} = \frac{6}{(\vartheta\Delta t)^2}(q_{t+\vartheta\Delta t} - q_t) - \frac{6\dot{q}_t}{\vartheta\Delta t} - 2\ddot{q}_t \quad (4.2.18)$$

The difference formulas in the Wilson theta algorithm are then given by

$$\dot{\mathbf{q}}_{t+\vartheta\Delta t} = \frac{3}{\vartheta\Delta t}(\mathbf{q}_{t+\vartheta\Delta t} - \mathbf{q}_t) - 2\dot{\mathbf{q}}_t - \frac{\ddot{\mathbf{q}}_t\vartheta\Delta t}{2} \quad (4.2.19)$$

$$\ddot{\mathbf{q}}_{t+\vartheta\Delta t} = \frac{6}{(\vartheta\Delta t)^2}(\mathbf{q}_{t+\vartheta\Delta t} - \mathbf{q}_t) - \frac{6\dot{\mathbf{q}}_t}{\vartheta\Delta t} - 2\ddot{\mathbf{q}}_t \quad (4.2.20)$$

We consider the equation of motion at time $t + \vartheta\Delta t$ to obtain solution for the displacements, velocities and acceleration at time $t + \Delta t$. Since the acceleration vary linearly, a linearly projected force vector is used such that

$$\mathbf{M}\ddot{\mathbf{q}}_{t+\vartheta\Delta t} + \mathbf{B}\dot{\mathbf{q}}_{t+\vartheta\Delta t} + \mathbf{K}\mathbf{q}_{t+\vartheta\Delta t} = \mathbf{Q}_{t+\vartheta\Delta t} \quad (4.2.21)$$

where

$$\mathbf{Q}_{t+\vartheta\Delta t} = \mathbf{Q}_t + \vartheta(\mathbf{Q}_{t+\Delta t} - \mathbf{Q}_t)$$

Substituting (4.2.19) and (4.2.20) into (4.2.21), we obtain

$$\begin{aligned} \left(\frac{6}{\vartheta^2\Delta t^2}\mathbf{M} + \frac{3}{\vartheta\Delta t}\mathbf{B} + \mathbf{K} \right) \mathbf{q}_{t+\vartheta\Delta t} &= \mathbf{Q}_{t+\vartheta\Delta t} + \left(\frac{6}{\vartheta^2\Delta t^2}\mathbf{M} + \frac{3}{\vartheta\Delta t}\mathbf{B} \right) \mathbf{q}_t + \\ &+ \left(\frac{6}{\vartheta\Delta t}\mathbf{M} + 2\mathbf{B} \right) \dot{\mathbf{q}}_t + \left(2\mathbf{M} + \frac{\vartheta\Delta t}{2}\mathbf{B} \right) \ddot{\mathbf{q}}_t \end{aligned} \quad (4.2.22)$$

We express the effective mass matrix $\bar{\mathbf{M}}$ and effective force vector $\bar{\mathbf{Q}}_{t+\vartheta\Delta t}$

$$\bar{\mathbf{M}} = \frac{6}{\vartheta^2\Delta t^2}\mathbf{M} + \frac{3}{\vartheta\Delta t}\mathbf{B} + \mathbf{K} \quad (4.2.23)$$

$$\bar{\mathbf{Q}}_{t+\vartheta\Delta t} = \mathbf{Q}_{t+\vartheta\Delta t} + \left(\frac{6}{\vartheta^2\Delta t^2}\mathbf{M} + \frac{3}{\vartheta\Delta t}\mathbf{B} \right) \mathbf{q}_t + \left(\frac{6}{\vartheta\Delta t}\mathbf{M} + 2\mathbf{B} \right) \dot{\mathbf{q}}_t + \left(2\mathbf{M} + \frac{\vartheta\Delta t}{2}\mathbf{B} \right) \ddot{\mathbf{q}}_t \quad (4.2.24)$$

By using of these equations we obtain

$$\mathbf{q}_{t+\vartheta\Delta t} = \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}_{t+\vartheta\Delta t} \quad (4.2.25)$$

The solution (4.2.25) yields $\mathbf{q}_{t+\vartheta\Delta t}$, which is then substituted in the following equations to obtain accelerations, velocities and displacements at $t + \Delta t$:

$$\ddot{\mathbf{q}}_{t+\Delta t} = \frac{6}{\vartheta^3\Delta t^2}(\mathbf{q}_{t+\vartheta\Delta t} - \mathbf{q}_t) - \frac{6}{\vartheta^2\Delta t}\dot{\mathbf{q}}_t + \left(1 - \frac{3}{\vartheta}\right)\ddot{\mathbf{q}}_t \quad (4.2.26)$$

$$\dot{\mathbf{q}}_{t+\Delta t} = \dot{\mathbf{q}}_t + \frac{\Delta t}{2}(\ddot{\mathbf{q}}_{t+\Delta t} + \ddot{\mathbf{q}}_t) \quad (4.2.27)$$

$$\mathbf{q}_{t+\Delta t} = \mathbf{q}_t + \Delta t\dot{\mathbf{q}}_t + \frac{\Delta t^2}{6}(\ddot{\mathbf{q}}_{t+\Delta t} + 2\ddot{\mathbf{q}}_t) \quad (4.2.28)$$

The method is proven to be unconditionally stable for values $\vartheta \geq 1,37$ for linear dynamic systems, but a value of 1,5 is often used for nonlinear problems. An anomaly of this method is that equilibrium is never satisfied at time $t + \Delta t$.

4.2.3 Newmark beta method

The Newmark integration method can be treated as an extension of the linear integration scheme. The method uses parameters α and β , which can be changed to suit the requirements of the problem at hand. The equations used are given

$$\dot{\mathbf{q}}_{t+\Delta t} = \dot{\mathbf{q}}_t + [(1-\alpha)\ddot{\mathbf{q}}_t + \alpha\ddot{\mathbf{q}}_{t+\Delta t}]\Delta t \quad (4.2.29)$$

$$\mathbf{q}_{t+\Delta t} = \mathbf{q}_t + \dot{\mathbf{q}}_t\Delta t + \left[\left(\frac{1}{2} - \beta\right)\ddot{\mathbf{q}}_t + \beta\ddot{\mathbf{q}}_{t+\Delta t}\right](\Delta t)^2 \quad (4.2.30)$$

α and β are parameters which are determined to obtain integration accuracy and stability. The effect of these parameters is the change the form of the variation of acceleration during the time interval Δt :

$\alpha = \frac{1}{2}$ and $\beta = 0$ the acceleration is constant and equal to \ddot{q}_t during each time interval Δt

$\alpha = \frac{1}{2}$ and $\beta = \frac{1}{8}$ the acceleration is constant from the beginning as \ddot{q}_t and then changes to

$\ddot{q}_{t+\Delta t}$ at the middle of the time interval Δt .

$\alpha = \frac{1}{2}$ and $\beta = \frac{1}{6}$ the acceleration varies linearly from \ddot{q}_t to $\ddot{q}_{t+\Delta t}$

$\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$ the acceleration remains constant at an average value of $(\ddot{q}_t + \ddot{q}_{t+\Delta t})/2$

The difference formulas in the Newmark beta algorithm are

$$\ddot{\mathbf{q}}_{t+\Delta t} = \frac{1}{\beta\Delta t^2}(\mathbf{q}_{t+\Delta t} - \mathbf{q}_t) - \frac{1}{\beta\Delta t}\dot{\mathbf{q}}_t - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{q}}_t \quad (4.2.31)$$

$$\dot{\mathbf{q}}_{t+\Delta t} = \frac{\alpha}{\beta\Delta t}(\mathbf{q}_{t+\Delta t} - \mathbf{q}_t) - \left(\frac{\alpha}{\beta} - 1\right)\dot{\mathbf{q}}_t - \Delta t\left(\frac{\alpha}{2\beta}\right)\ddot{\mathbf{q}}_t \quad (4.2.32)$$

Substituting (4.2.31) and (4.2.32) into the equation of motion at time $t + \Delta t$

$$\mathbf{M}\ddot{\mathbf{q}}_{t+\Delta t} + \mathbf{B}\dot{\mathbf{q}}_{t+\Delta t} + \mathbf{K}\mathbf{q}_{t+\Delta t} = \mathbf{Q}_{t+\Delta t}$$

We express the effective mass and effective force vector:

$$\bar{\mathbf{M}} = \frac{1}{\beta\Delta t^2}\mathbf{M} + \frac{\alpha}{\beta\Delta t}\mathbf{B} + \mathbf{K} \quad (4.2.33)$$

$$\begin{aligned} \bar{\mathbf{Q}}_{t+\Delta t} = & \mathbf{Q}_{t+\Delta t} + \left[\left(\frac{1}{2\beta} - 1\right)\mathbf{M} + \Delta t\left(\frac{\alpha}{2\beta} - 1\right)\mathbf{B} \right] \ddot{\mathbf{q}}_t + \left[\frac{1}{\beta\Delta t}\mathbf{M} + \left(\frac{\alpha}{\beta} - 1\right)\mathbf{B} \right] \dot{\mathbf{q}}_t + \\ & + \left(\frac{1}{\beta\Delta t^2}\mathbf{M} + \frac{\alpha}{\beta\Delta t}\mathbf{B} \right) \mathbf{q}_t \end{aligned} \quad (4.2.34)$$

By using of these formulas it is possible to determine

$$\mathbf{q}_{t+\Delta t} = \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}_{t+\Delta t} \quad (4.2.35)$$

(4.2.35) yields $\mathbf{q}_{t+\Delta t}$ which is then substituted in (4.2.31) and (4.2.32) to obtain velocities and accelerations at $t + \Delta t$.

The important features of this method are that for linear systems the amplitude of mode is conserved, and the response is unconditionally stable provided that $\alpha \geq \frac{1}{2}$ and

$\beta \geq 0,25(\alpha + 0,5)^2$. However, the $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$ give the largest truncation error in the frequency of the response.

5. TUNING OF MECHANICAL SYSTEMS

The process changing the masses and stiffness of mechanical systems to obtain the required natural frequencies and natural modes, is called *tuning* of mechanical systems.

For simplicity we will consider free non damped mechanical system described by its mass, stiffness and geometrical parameters. These elements generate *vector of tuning parameters* or *tuned vector*.

$$\mathbf{p}^T = [p_1, p_2, \dots, p_{s_p}]$$

The eigenvalues $\lambda_i = \Omega_i^2$ and natural vectors, normed by mass matrix, expresses *vector of tuned parameters*

$$\mathbf{l}^T = [\Omega_1^2, \Omega_2^2, \dots, \Omega_n^2, \mathbf{v}_1^T, \mathbf{v}_2^T, \dots, \mathbf{v}_n^T]$$

Usually, we do not change all elements of the tuned vector but only some of them. In such case is defined the *selecting vector* $\mathbf{j} = [j_i]$ of order $k < n$. The selecting vector determine the elements of tuning vector to them we specify certain values. So arise the *reduced tuned vector* $\mathbf{l}_r^T = [l_{r1}, l_{r2}, \dots, l_{rk}]$. Often we require change of natural frequencies. In such case we speak about *spectral tuning*. If modal vectors are to be changed we speak about *modal tuning*.

The *vector of required tuning values* is signed \mathbf{l}^* which is of order k . To reach this vector we change only some elements of the tuning vector. So arise the *reduced vector of tuning parameters* \mathbf{p}_r , which is defined by the *selecting vector of tuning parameters*

$$\mathbf{i} = [i_1, i_2, \dots, i_s]$$

with s elements. In next we omit the word reduced as well the index r by both vectors – tuning and tuned. The required tuned values depend on tuning parameters $\mathbf{l} = \mathbf{l}(\mathbf{p})$.

5.1 The method of successive linear approximations

The tuning is mathematical formulate as finding the vector \mathbf{p}^* so that

$$\mathbf{l}(\mathbf{p}^*) = \mathbf{l}^* \tag{5.1.1}$$

The tuning process do not change the tuning parameters, which are not involved into the reduced vector of tuning parameters \mathbf{p}_r . However, the tuned parameters that may not be changed must be involved in the tuned vector \mathbf{l} .

If the vector of tuned parameters $\mathbf{l}(\mathbf{p})$ is defined in the surroundings of outgoing point p_0 it is possible every function $\mathbf{l}_i(\mathbf{p})$ expand into the Taylor series

$$\begin{aligned}
l_i(\mathbf{p}) = & l_i(\mathbf{p}_0) + \sum_{i_1=1}^s \frac{\partial l_i(\mathbf{p}_0)}{\partial p_{i_1}} (p_{i_1} - p_{i_1}^{(0)}) + \frac{1}{2} \sum_{i_1=1}^{s_p} \sum_{i_2=1}^{s_p} \frac{\partial^2 l_i(\mathbf{p}_0)}{\partial p_{i_1} \partial p_{i_2}} (p_{i_1} - p_{i_1}^{(0)}) (p_{i_2} - p_{i_2}^{(0)}) + \\
& \frac{1}{3!} \sum_{i_1=1}^{s_p} \sum_{i_2=1}^{s_p} \sum_{i_3=1}^{s_p} \frac{\partial^3 l_i(\mathbf{p}_0)}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}} (p_{i_1} - p_{i_1}^{(0)}) (p_{i_2} - p_{i_2}^{(0)}) (p_{i_3} - p_{i_3}^{(0)}) + \dots
\end{aligned} \tag{5.1.2}$$

We will take into account the linear restitution (we use only two first terms of (5.1.2)). We introduce the gradient vector of the function $l_i(\mathbf{p}_0)$:

$$\mathit{grad} l_i(\mathbf{p}_0) = \left[\frac{\partial l_i(\mathbf{p}_0)}{\partial p_1}, \dots, \frac{\partial l_i(\mathbf{p}_0)}{\partial p_s} \right]^T \tag{5.1.3}$$

By using (5.1.3) in (5.1.2) we may write

$$l_i(\mathbf{p}) \doteq l_i(\mathbf{p}_0) + \mathit{grad}^T l_i(\mathbf{p}_0) (\mathbf{p} - \mathbf{p}_0) \quad \text{for } i = 1, 2, \dots, k \tag{5.1.4}$$

We use the *Jacobi matrix of notation*, which is also called the *tuning matrix*

$$\mathbf{L}(\mathbf{p}_0) = \begin{bmatrix} \mathit{grad}^T l_1(\mathbf{p}_0) \\ \vdots \\ \mathit{grad}^T l_k(\mathbf{p}_0) \end{bmatrix} = \left[\frac{\partial l_i(\mathbf{p}_0)}{\partial p_j} \right] \quad \text{for } i = 1, 2, \dots, k \quad j = 1, 2, \dots, s$$

The notation of (5.1.4) is possible to simplify

$$\mathbf{l}(\mathbf{p}) \doteq \mathbf{l}(\mathbf{p}_0) + \mathbf{L}(\mathbf{p}_0) (\mathbf{p} - \mathbf{p}_0) \tag{5.1.5}$$

The tuning matrix is generally rectangular (k, s). Its elements in i – row and j – column express the rate of change tuned value on the change of tuning parametr, therefore it is possible to call it the *sensitivity matrix*. If the tuning matrix is regular and for its majority is valid

$$h\mathbf{L}(\mathbf{p}_0) = \min(k, s)$$

Then exists for $k \geq s$ the left hand side inverse matrix

$$\mathbf{L}^L(\mathbf{p}_0) = \left(\mathbf{L}^T(\mathbf{p}_0) \mathbf{L}(\mathbf{p}_0) \right)^{-1} \mathbf{L}^T(\mathbf{p}_0)$$

For $k \leq s$ exists right hand side inverse matrix

$$\mathbf{L}^P(\mathbf{p}_0) = \mathbf{L}^T(\mathbf{p}_0) \left(\mathbf{L}(\mathbf{p}_0) \mathbf{L}^T(\mathbf{p}_0) \right)^{-1}$$

By using of these equations it is possible determine from (5.1.5)

$$\mathbf{p} \doteq \mathbf{p}_0 + \mathbf{L}^+(\mathbf{p}_0) (\mathbf{l}(\mathbf{p}) - \mathbf{l}(\mathbf{p}_0)) \tag{5.1.6}$$

\mathbf{L}^+ is right hand side or left hand side matrix.

If we want solve (5.1.1) we get by using (5.1.5) and (5.1.6)

$$\mathbf{p} \doteq \mathbf{p}_0 + \mathbf{L}^+(\mathbf{p}_0) (\mathbf{l}^* - \mathbf{l}(\mathbf{p}_0)) \tag{5.1.7}$$

This expression is loaded by an error, because

1. The linear consideration (5.1.5) is by non-linear systems approximate
2. If $r > s$ the solution (5.1.6) is only the best approximation

To diminish the error we consider (5.1.7) as iteration and we use the expression

$$\mathbf{p}_{l+1} = \mathbf{p}_l + \mathbf{L}^+(\mathbf{p}_l)(\mathbf{I}^* - \mathbf{I}(\mathbf{p}_l)) \quad \text{for } l = 1, 2, \dots \quad (5.1.8)$$

From the initial value \mathbf{p}_0 we determine the first member of the series \mathbf{p}_l and successively the next terms. If the series converge we sign the limit \mathbf{p}^* as the result. The basic requirement of solution (5.1.1) is to obtain the best values of tuned values. The tuning process is finished if the following condition is fulfilled

$$\sum_{j=1}^k g_j \left[1 - \frac{\mathbf{I}_j(\mathbf{p}_l)}{\mathbf{I}_j^*} \right]^2 \leq \varepsilon_1 \quad (5.1.9)$$

ε_1 is chosen small positive number, which define the allowed error of tuned values. g_i are positive weight coefficients, that allowed to prefer some of tuned parameters. (5.1.9) used relative errors. The square values are used because it does not depend on the sign of the difference. By accurate solution would be the left hand side of equation (5.1.9) equal to zero. In all other cases is positive.

In some cases the exact solution does not exist. For very small values of ε_1 , (5.1.9) will be never fulfilled, and even it can coverage. Therefore we introduce other criterion to stop the calculation

$$\sum_{j=1}^{s_p} \left[1 - \frac{p_j^{(l+1)}}{p_j^{(l)}} \right]^2 \leq \varepsilon_2 \quad (5.1.10)$$

ε_2 is again relative error of tuned parameters. If the process does not converge it is necessary to stop the calculation after a fix given number k_0 of iteration steps.

5.2 Dynamic sensitivity

The aim of sensitivity analysis is to obtain quantitative information about the sensitivity of structural natural frequencies and natural and natural modes to variations of tuning parameters such as spring stiffness, elasticity coefficients of materials, concentrated masses, distribute masses, cross section area etc. Dynamic sensitivity is defined as the ratio of natural frequencies or natural vectors to the unit change of tuning parameters.

The sensitivity analysis is becoming more valuable in dynamic structural systems for several purposes:

- to get a better knowledge of the sensitivity of a structure to slight modifications
- to obtain derivatives for the dynamic optimization and tuning of a structure by mathematical programming methods
- to obtain derivatives for the updating of a dynamic model which aims to match numerical and experimental results

To derive dynamic sensitivity and also the coefficients of tuning matrix given by (5.1.4)

it is necessary to derive $\frac{\partial \lambda_i}{\partial p_j}$ and $\frac{\partial v_{ki}}{\partial p_j}$. Here is $\lambda_i = \Omega_i^2$, v_{ki} is k – element of i – natural vector,

p_j is a tuning parameter. As was derived the free non-damped system is described by the equation

$$(\mathbf{K} - \lambda_i \mathbf{M}) \mathbf{v}_i = \mathbf{0} \quad (5.2.1)$$

Or in the form

$$\mathbf{K} \mathbf{v}_i = \lambda_i \mathbf{M} \mathbf{v}_i \quad (5.2.2)$$

We suppose the normalization of the vector $\mathbf{v}_i^T \mathbf{M} \mathbf{v}_i = 1$ and multiply (5.2.2) by \mathbf{v}_i^T from left hand side we obtain a simple form

$$\mathbf{v}_i^T \mathbf{K} \mathbf{v}_i = \lambda_i \quad (5.2.3)$$

The derivative of (5.2.2) over p_i we get

$$\frac{\partial \mathbf{K}}{\partial p_j} \mathbf{v}_i + \mathbf{K} \frac{\partial \mathbf{v}_i}{\partial p_j} = \frac{\partial \lambda_i}{\partial p_j} \mathbf{M} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{M}}{\partial p_j} \mathbf{v}_i + \lambda_i \mathbf{M} \frac{\partial \mathbf{v}_i}{\partial p_j}$$

After multiplying of this equation by \mathbf{v}_i^T from left hand side and using the previous normalization we obtain

$$\frac{\partial \lambda_i}{\partial p_j} = \mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial p_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_i + \mathbf{v}_i^T (\mathbf{K} - \lambda_i \mathbf{M}) \frac{\partial \mathbf{v}_i}{\partial p_j} \quad (5.2.4)$$

After transposition of (5.2.1) is

$$\mathbf{v}_i^T (\mathbf{K} - \lambda_i \mathbf{M}) = \mathbf{0}^T$$

When we apply this equation on (5.2.4) we get

$$\frac{\partial \lambda_i}{\partial p_j} = \mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial p_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_i \quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, s \quad (5.2.5)$$

(5.2.5) represents the sensibility of eigenvalue λ_i to the change of tuning parameter p_j .

For next solution we suppose that $\frac{\partial \mathbf{v}_k}{\partial p_j}$ is a linear combination of natural vectors

$$\frac{\partial \mathbf{v}_k}{\partial p_j} = \sum_{i=1}^n a_{ki}^{(j)} \mathbf{v}_i \quad (5.2.6)$$

After substituting (5.2.6) into (5.2.4) we obtain after arrangement the equation for non-diagonal elements:

$$\mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial p_j} - \lambda_k \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_k + \sum_{l=1}^n a_{ki}^{(j)} \mathbf{v}_i^T (\mathbf{K} - \lambda_k \mathbf{M}) \mathbf{v}_l = 0 \quad (5.2.7)$$

With respect to (5.2.3) we may write

$$\mathbf{v}_i^T \mathbf{M} \mathbf{v}_l = \delta_{il} \quad \mathbf{v}_i^T \mathbf{K} \mathbf{v}_l = \lambda_i \delta_{il} \quad (5.2.8)$$

The Kronecker coefficient has the values

$$\begin{aligned} \delta_{il} &= 1 \quad \text{pro } i = l \\ \delta_{il} &= 0 \quad \text{pro } i \neq l \end{aligned}$$

Now we can write (5.2.7) in the form

$$\mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial p_j} - \lambda_k \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_k + a_{ki}^{(j)} (\lambda_i - \lambda_k) = 0$$

From this equation we obtain

$$a_{ki}^{(j)} = \frac{1}{\lambda_k - \lambda_i} \mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial p_j} - \lambda_k \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_k \quad (5.2.9)$$

If some of eigenvalue are equal we will work as shown in next process

We make derivative of the expression for normalization

$$\frac{\partial \mathbf{v}_i^T}{\partial p_j} \mathbf{M} \mathbf{v}_i + \mathbf{v}_i^T \frac{\partial \mathbf{M}}{\partial p_j} \mathbf{v}_i + \mathbf{v}_i^T \mathbf{M} \frac{\partial \mathbf{v}_i}{\partial p_j} = 0$$

Each summand in this equation is scalar. The first summand is obtained by transposition of the last one and we can write

$$2 \mathbf{v}_i^T \mathbf{M} \frac{\partial \mathbf{v}_i}{\partial p_j} = - \mathbf{v}_i^T \frac{\partial \mathbf{M}}{\partial p_j} \mathbf{v}_i$$

When we substitute in this equation (5.2.6), then

$$a_{ii} = - \frac{1}{2} \mathbf{v}_i^T \frac{\partial \mathbf{M}}{\partial p_j} \mathbf{v}_i \quad (5.2.10)$$

Substituting the coefficient given by (5.2.9) and (5.2.10) in (5.2.6) we obtain the expression for determination of the sensitivity natural vectors:

$$\frac{\partial \mathbf{v}_k}{\partial p_j} = \sum_{\substack{i=1 \\ i \neq k}}^n \left[\frac{1}{\lambda_k - \lambda_i} \mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial p_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_k \mathbf{v}_i \right] - \frac{1}{2} \mathbf{v}_k^T \frac{\partial \mathbf{M}}{\partial p_j} \mathbf{v}_k \mathbf{v}_k \quad (5.2.11)$$

5.2.1 The tuning process

The starting state is the designed mechanical model with given mass, stiffness and geometrical parameters. In such case the mass matrix \mathbf{M} and stiffness matrix \mathbf{K} are known. Then we must know which elements of the tuned vector will change what values they have to reach. The tuning process is possible to describe by following algorithm:

1. We determine the natural frequencies Ω_i , and natural vectors \mathbf{v}_i of the designed model.

We provide the normalization over the mass matrix \mathbf{M} .

2. With respect of tuned requirements we determine the selecting vector of tuned parameters $\mathbf{i} = [i_j]_{j=1}^k$ and so will be determined the tuned vector \mathbf{l} .

3. We make the sensitivity analysis of the designed model. By this way the starting values of tuning matrix \mathbf{L} are given. This matrix is a column matrix of order s , which involves all structure parameters, which may be changed.

4. We determine the selecting vector of tuning parameters $\mathbf{j} = [j_i]_{i=1}^s$. So the tuning vector \mathbf{p} is given. The number of tuning parameters we preferably choose equal to the number of requirements k . In such case $\mathbf{L}^* = \mathbf{L}^{-1}$ and the solution is exact. The selection of tuning parameters is made with respect of the results of the sensitivity analysis. The sensitivity determines the absolute values of the tuning matrix and the tuning parameters correspond to the columns of this matrix. The measure of sensitivity of tuning parameters is the sum of absolute values of the column matrix $\mathbf{L} = [l_{ij}]$ for $i = 1, 2, \dots, k; j = 1, 2, \dots, s$. Whereby the sum $\left(c_j = \sum_{i=1}^k |l_{ij}| \right)$ is greater so the sensitivity is greater.

5. We choose the relative errors of tuned parameters ε_1 and tuning parameters ε_2 , the greatest number of iterations k_0 , eventually the weight coefficients of tuned values. By using of simple arithmetic we do not choose relative errors greater than 10^{-8} . In the

case $s \geq k$ we choose $\varepsilon_1 > \varepsilon_2$ approximately of two orders. The number k_0 is not necessary be greater than 10.

6. We decide about using of admissible region and of the possibility of a shortening of the step $\|\Delta \mathbf{p}\|$. Using of admissible region is necessary from the physical point of view (tuning parameters can not be negative). With respect to this we determine the lower and upper *bar*. If the upper bar may grow to infinity we choose it of some order greater than the starting value.
7. We start the own tuning of the system according the equation (5.1.8)
8. We check the results of tuning process and its compilation. If the process divergates we choose smaller value of the step and we repeat the calculation.
9. We calculate natural frequencies and modal vectors of the mechanical system with new parameters. We compare the results with given requirements.

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